PROBLEMA 1

Probar que si $f(t)$ es una función continua en el punto $t = a$ entonces

$$f(t)\delta(t-a) = f(a)\delta(t-a)$$

SOLUCIÓN:

Sea $\phi(t) \in \mathcal{C}^\infty(\mathbb{R})$ una función prueba:

$$\langle f(t)\delta(t-a), \phi(t)\rangle = \langle \delta(t-a), f(t)\phi(t)\rangle = f(a)\phi(a) = f(a)\langle \delta(t-a), \phi(t)\rangle =$$

$$= \langle f(a)\delta(t-a), \phi(t)\rangle \quad \Rightarrow \quad f(t)\delta(t-a) = f(a)\delta(t-a)$$

PROBLEMA 2

Demuestre que

$$\frac{d}{dx}\big((1+x^2)\delta(x-1)\big) = 2\delta'(x-1)$$

SOLUCIÓN:

Sea $\phi(x) \in \mathcal{C}^\infty(\mathbb{R})$ una función prueba:

$$\langle\frac{d}{dx}\big((1+x^2)\delta(x-1)\big), \phi(x)\rangle = \langle(1+x^2)\delta(x-1), -\phi'(x)\rangle =$$

$$\langle \delta(x-1), -(1+x^2)\phi'(x)\rangle = -(1+x^2)\phi'(x)\bigg|_{x=1} = -\left(1+1^2\right)\phi'(1) = -2\phi'(1) =$$

$$-2\langle \delta(x-1), \phi'(x)\rangle = -2\langle -\delta'(x-1), \phi(x)\rangle = \langle 2\delta'(x-1), \phi(x)\rangle$$

PROBLEMA 3

Muestre que $(x^2+1)\delta'(x) = \delta'(x)$ en el espacio de funciones de prueba $\mathcal{C}^\infty(\mathbb{R})$.

SOLUCIÓN:

Sea $\phi(x) \in \mathcal{C}^\infty(\mathbb{R})$ una función prueba:

$$\langle (x^2+1)\delta'(x), \phi(x)\ \rangle = \langle \delta'(x), (x^2+1)\phi(x)\ \rangle = \langle \delta(x), -\frac{d}{dx}\big((x^2+1)\phi(x)\big)\ \rangle =$$

$$= \langle \delta(x), -2x\phi(x) - (x^2+1)\phi'(x)\rangle = -2\langle \delta(x), x\phi(x)\rangle - \langle \delta(x), (x^2+1)\phi'(x)\rangle =$$

$$-2x\phi(x)\bigg|_{x=0} - (x^2+1)\phi'(x)\bigg|_{x=0} = -2\cdot 0\cdot \phi(0) - (0^2+1)\phi'(0) =$$

$$= -\phi'(0) = -\langle \delta(x), \phi'(x)\rangle = -\langle -\delta'(x), \phi(x)\rangle = \langle \delta'(x), \phi(x)\rangle$$

PROBLEMA 4

Exprese $\sqrt{1+x}\,\delta''(x)$ en la forma

$$c_0\delta(x)+c_1\delta'(x)+c_2\delta''(x)$$

En el intervalo $-1\leq x<\infty$.

SOLUCIÓN:

Sea $f(x)=\sqrt{1+x}$ y $\phi(x)\in C^{\infty}(\mathbb{R})$ una función prueba

$$\langle f(x)\delta''(x),\phi(x)\rangle=\langle\delta''(x),f(x)\phi(x)\rangle=\langle\delta(x),\frac{d^2}{dx^2}\big(f(x)\phi(x)\big)\rangle=$$

$$=\langle\delta(x),f''(x)\phi(x)+2f'(x)\phi'(x)+f(x)\phi''(x)\rangle=$$

$$=\langle\delta(x),f''(x)\phi(x)\rangle+2\langle\delta(x),f'(x)\phi'(x)\rangle+\langle\delta(x),f(x)\phi''(x)\rangle=$$

$$=f''(0)\phi(0)+2f'(0)\phi'(0)+f(0)\phi''(0)=$$

$$=f''(0)\langle\delta(x),\phi(x)\rangle+2f'(0)\langle\delta(x),\phi'(x)\rangle+f(0)\langle\delta(x),\phi''(x)\rangle=$$

$$=f''(0)\langle\delta(x),\phi(x)\rangle+2f'(0)(-1)\langle\delta'(x),\phi(x)\rangle+f(0)(-1)^2\langle\delta''(x),\phi(x)\rangle=$$

$$=\langle f''(0)\delta(x),\phi(x)\rangle+\langle-2f'(0)\delta'(x),\phi(x)\rangle+\langle f(0)\delta''(x),\phi(x)\rangle=$$

$$=\langle f''(0)\delta(x)-2f'(0)\delta'(x)+f(0)\delta''(x),\phi(x)\rangle$$

Encontrando los valores de $f(0)$, $f'(0)$, $f''(0)$:

$$f(x)=\sqrt{1+x}\Rightarrow f(0)=\sqrt{1+0}=1$$

$$f'(x)=\frac{1}{2\sqrt{1+x}}\Rightarrow f'(0)=\frac{1}{2\sqrt{1+0}}=\frac{1}{2}$$

$$f''(x)=-\frac{1}{4}\cdot\frac{1}{(1+x)^{\frac{3}{2}}}\Rightarrow f''(0)=-\frac{1}{4}\cdot\frac{1}{(1+0)^{\frac{3}{2}}}=-\frac{1}{4}$$

Y sustituyendo en la expresión deducida anteriormente

$$\langle\sqrt{1+x}\,\delta''(x),\phi(x)\rangle=\langle-\frac{1}{4}\delta(x)-\delta'(x)+\delta''(x),\phi(x)\rangle$$

Así

$$\sqrt{1+x}\,\delta''(x)=-\frac{1}{4}\delta(x)-\delta'(x)+\delta''(x)$$

PROBLEMA 5

Muestre que

$$(1+\cos x)\delta''(x-\pi)=\delta(x-\pi)$$

SOLUCIÓN:

Sea $f(x)=1+\cos x$ y $\phi(x)\in\mathcal{C}^{\infty}(\mathbb{R})$ una función prueba

$$\langle f(x)\delta''(x-\pi),\phi(x)\rangle=\langle\delta''(x-\pi),f(x)\phi(x)\rangle=\langle\delta(x-\pi),\frac{d^2}{dx^2}\big(f(x)\phi(x)\big)\rangle=$$

$$=\langle\delta(x-\pi),f''(x)\phi(x)+2f'(x)\phi'(x)+f(x)\phi''(x)\rangle=$$

$$=\langle\delta(x-\pi),f''(x)\phi(x)\rangle+2\langle\delta(x-\pi),f'(x)\phi'(x)\rangle+\langle\delta(x-\pi),f(x)\phi''(x)\rangle=$$

$$=f''(\pi)\phi(\pi)+2f'(\pi)\phi'(\pi)+f(\pi)\phi''(\pi)=$$

$$=f''(\pi)\langle\delta(x-\pi),\phi(x)\rangle+2f'(\pi)\langle\delta(x-\pi),\phi'(x)\rangle+f(\pi)\langle\delta(x-\pi),\phi''(x)\rangle=$$

$$=f''(\pi)\langle\delta(x-\pi),\phi(x)\rangle+2f'(\pi)(-1)\langle\delta'(x-\pi),\phi(x)\rangle+f(\pi)(-1)^2\langle\delta''(x-\pi),\phi(x)\rangle=$$

$$=\langle f''(\pi)\delta(x-\pi),\phi(x)\rangle+\langle -2f'(\pi)\delta'(x-\pi),\phi(x)\rangle+\langle f(\pi)\delta''(x-\pi),\phi(x)\rangle=$$

$$=\langle f''(\pi)\delta(x)-2f'(\pi)\delta'(x)+f(\pi)\delta''(x),\phi(x)\rangle$$

Encontrando los valores de $f(\pi)$, $f'(\pi)$, $f''(\pi)$:

$$f(x)=1+\cos x \quad\Rightarrow\quad f(\pi)=1+\cos(\pi)=1-1=0$$

$$f'(x)=0-\operatorname{sen}x \quad\Rightarrow\quad f'(\pi)=-\operatorname{sen}(\pi)=0$$

$$f''(x)=-\cos x \quad\Rightarrow\quad f''(\pi)=-\cos(\pi)=-(-1)=1$$

Y sustituyendo en la expresión deducida anteriormente

$$\langle(1+\cos x)\,\delta''(x-\pi),\phi(x)\rangle=\langle 1\cdot\delta(x-\pi)-2\cdot 0\cdot\delta'(x-\pi)+0\cdot\delta''(x-\pi),\phi(x)\rangle$$
$$=\langle\delta(x-\pi),\phi(x)\rangle$$

Así

$$(1+\cos x)\,\delta''(x-\pi)=\delta(x-\pi)$$

PROBLEMA 6

Halle las constantes a, b y $c \in \mathbb{R}$ tales que se cumpla la igualdad

$$\cot x\, \delta''\left(x-\frac{\pi}{4}\right) = a\delta\left(x-\frac{\pi}{4}\right) + b\delta\,'\left(x-\frac{\pi}{4}\right) + c\delta\,''\left(x-\frac{\pi}{4}\right)$$

SOLUCIÓN:

Sea $f(x) = \cot x$ y $\phi(x) \in \mathcal{C}^{\infty}(\mathbb{R})$ una función prueba y $x_0 = \pi/4$

$$\langle f(x)\delta''(x-x_0), \phi(x)\rangle = \langle \delta''(x-x_0), f(x)\phi(x)\rangle = \langle \delta(x-x_0), \frac{d^2}{dx^2}\big(f(x)\phi(x)\big)\rangle =$$

$$= \langle \delta(x-x_0), f''(x)\phi(x) + 2f'(x)\phi'(x) + f(x)\phi''(x)\rangle =$$

$$= \langle \delta(x-x_0), f''(x)\phi(x)\rangle + 2\langle \delta(x-x_0), f'(x)\phi'(x)\rangle + \langle \delta(x-x_0), f(x)\phi''(x)\rangle =$$

$$= f''(x_0)\phi(x_0) + 2f'(x_0)\phi'(x_0) + f(x_0)\phi''(x_0) =$$

$$= f''(x_0)\langle \delta(x-x_0), \phi(x)\rangle + 2f'(x_0)\langle \delta(x-x_0), \phi'(x)\rangle + f(x_0)\langle \delta(x-x_0), \phi''(x)\rangle =$$

$$= f''(x_0)\langle \delta(x-x_0), \phi(x)\rangle + 2f'(x_0)(-1)\langle \delta'(x-x_0), \phi(x)\rangle + f(x_0)(-1)^2\langle \delta''(x-x_0), \phi(x)\rangle$$

$$= \langle f''(x_0)\delta(x-x_0), \phi(x)\rangle + \langle -2f'(x_0)\delta'(x-x_0), \phi(x)\rangle + \langle f(x_0)\delta''(x-x_0), \phi(x)\rangle =$$

$$= \langle f''(x_0)\delta(x-x_0) - 2f'(x_0)\delta'(x-x_0) + f(x_0)\delta''(x-x_0), \phi(x)\rangle$$

Encontrando los valores de $f(x_0)$, $f'(x_0)$, $f''(x_0)$:

$$f(x) = \cot x \quad \Rightarrow \quad f\left(\frac{\pi}{4}\right) = \cot\left(\frac{\pi}{4}\right) = 1$$

$$f'(x) = -\csc^2 x \quad \Rightarrow \quad f'\left(\frac{\pi}{4}\right) = -\csc^2\left(\frac{\pi}{4}\right) = -\left(\sqrt{2}\right)^2 = -2$$

$$f''(x) = -2\csc x\,(-\csc x \cot x) = 2\csc^2 x \cot x \quad \Rightarrow \quad f''\left(\frac{\pi}{4}\right) = 2\csc^2\left(\frac{\pi}{4}\right)\cot\left(\frac{\pi}{4}\right) = 4$$

Y sustituyendo en la expresión deducida anteriormente

$$\langle \cot x\, \delta''\left(x-\frac{\pi}{4}\right), \phi(x)\rangle = \langle -4\delta\left(x-\frac{\pi}{4}\right) - 2\cdot(-2)\delta'\left(x-\frac{\pi}{4}\right) + 1\cdot\delta''\left(x-\frac{\pi}{4}\right), \phi(x)\rangle$$
$$= \langle -4\delta\left(x-\frac{\pi}{4}\right) + 2\delta'\left(x-\frac{\pi}{4}\right) + \delta''\left(x-\frac{\pi}{4}\right), \phi(x)\rangle$$

Así

$$\cot x\, \delta''\left(x-\frac{\pi}{4}\right) = 4\delta\left(x-\frac{\pi}{4}\right) + 2\delta'\left(x-\frac{\pi}{4}\right) + \delta''\left(x-\frac{\pi}{4}\right)$$

$$a = b = 4 \quad , \quad c = 1$$

PROBLEMA 7

Expresar la distribución

$$\operatorname{sen} x \frac{d}{dx}\big(\operatorname{sen}(3x)\,\delta'(x)\big)$$

Como combinación lineal de $\delta(x)$, $\delta'(x)$ y $\delta''(x)$.

SOLUCIÓN:

Sea $\phi(x) \in \mathcal{C}^{\infty}(\mathbb{R})$ una función prueba

$$\langle \operatorname{sen} x \frac{d}{dx}\big(\operatorname{sen}(3x)\,\delta'(x)\big), \phi(x)\rangle = \langle \frac{d}{dx}\big(\operatorname{sen}(3x)\,\delta'(x)\big), \operatorname{sen} x\,\phi(x)\rangle$$

$$= (-1)^1 \langle \operatorname{sen}(3x)\,\delta'(x), \frac{d}{dx}\big(\operatorname{sen} x\,\phi(x)\big)\rangle = -\langle \operatorname{sen}(3x)\,\delta'(x), \phi(x)\cos x + \phi'(x)\operatorname{sen} x\rangle =$$

$$= -\langle \delta'(x), \phi(x)\underbrace{\operatorname{sen}(3x)\cos x}_{g(x)} + \phi'(x)\underbrace{\operatorname{sen}(3x)\operatorname{sen} x}_{h(x)}\rangle = -\langle \delta'(x), \phi(x)g(x) + \phi'(x)h(x)\rangle$$

$$= -(-1)^1 \langle \delta(x), \frac{d}{dx}\big(\phi(x)g(x) + \phi'(x)h(x)\big)\rangle =$$

$$= \langle \delta(x), \phi'(x)g(x) + \phi(x)g'(x) + \phi''(x)h(x) + \phi'(x)h'(x)\rangle =$$

$$= \phi'(0)g(0) + \phi(0)g'(0) + \phi''(0)h(0) + \phi'(0)h'(0)$$

Hallando los valores de $g(0), g'(0), h(0)$ y $h'(0)$

$$g(x) = \operatorname{sen}(3x)\cos x \quad \Rightarrow \quad g(0) = \operatorname{sen}(0)\cos(0) = 0$$

$$g'(x) = 3\cos(3x)\cos x - \operatorname{sen}(3x)\operatorname{sen} x \quad \Rightarrow \quad g'(0) = 3\cos(0)\cos(0) - 0 = 3$$

$$h(x) = \operatorname{sen}(3x)\operatorname{sen} x \quad \Rightarrow \quad h(0) = \operatorname{sen}(0)\operatorname{sen}(0) = 0$$

$$h'(x) = 3\cos(3x)\operatorname{sen} x + \operatorname{sen}(3x)\cos x \quad \Rightarrow \quad h'(0) = 3\cos(0)\operatorname{sen}(0) + \operatorname{sen}(0)\cos(0) = 0$$

Sustituyendo estos valores

$$\langle \operatorname{sen} x \frac{d}{dx}\big(\operatorname{sen}(3x)\,\delta'(x)\big), \phi(x)\rangle = \phi'(0)\cdot 0 + \phi(0)\cdot 3 + \phi''(0)\cdot 0 + \phi'(0)\cdot 0 =$$

$$= 3\phi(0) = 3\langle \delta(x), \phi(x)\rangle = \langle 3\delta(x), \phi(x)\rangle$$

Por lo tanto si

$$\langle \operatorname{sen} x \frac{d}{dx}\big(\operatorname{sen}(3x)\,\delta'(x)\big), \phi(x)\rangle = \langle 3\delta(x), \phi(x)\rangle$$

Entonces

$$\operatorname{sen} x \frac{d}{dx}\big(\operatorname{sen}(3x)\,\delta'(x)\big) = 3\delta(x)$$

PROBLEMA 8

Sea $f\colon \mathbb{R} \to \mathbb{R}$ la función definida por

$$f(x) = (1-x)1_{(0,1)}(x)$$

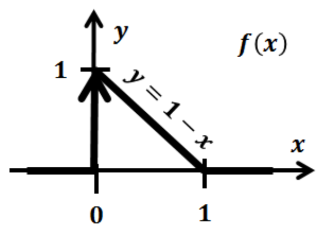
(a) Calcule $f''_{gen}(x)$

(b) Sea $\lambda > 0$. Use la parte (a) para calcular la siguiente integral

$$I(\lambda) = \int_0^1 (1-x)e^{\lambda x}dx$$

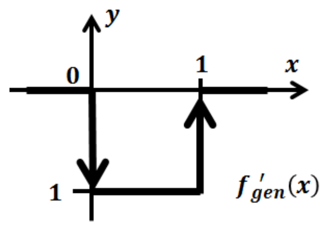
SOLUCIÓN:

$$1_{(0,1)}(x) = \begin{cases} 1 \ \text{ si } \ 0 \le x \le 1 \\ 0 \qquad\quad \text{si no} \end{cases}$$



$$f(x) = (1-x)1_{(0,1)}(x) = \begin{cases} 1-x \ \text{ si } \ 0 \le x \le 1 \\ 0 \qquad\quad \text{si no} \end{cases}$$

Calculando la primera derivada generalizada:



$$f'_{gen}(x) = \begin{cases} -1 \ \text{ si } \ 0 \le x \le 1 \\ 0 \qquad\quad \text{si no} \end{cases} + \delta(x)$$

Calculando la segunda derivada generalizada:

$$f''_{gen}(x) = 0 + \delta'(x) - \delta(x) + \delta(x-1) = \delta'(x) - \delta(x) + \delta(x-1)$$

$$I(\lambda) = \int_0^1 (1-x)e^{\lambda x}dx = \int_{-\infty}^{\infty} (1-x)1_{(0,1)}(x)\underbrace{e^{\lambda x}}_{\phi(x)}dx = \int_{-\infty}^{\infty} f(x)\,\phi(x)dx = \langle f(x), \phi(x)\rangle$$

Como $\phi(x) = e^{\lambda x} \in C^{\infty}(\mathbb{R})$ es una función prueba que satisface la ecuación diferencial

$$\phi''(x) - \lambda^2\phi(x) = 0 \quad \Rightarrow \quad \phi(x) = \frac{1}{\lambda^2}\phi''(x)$$

Sustituyendo esta expresión en el producto interno

$$I(\lambda) = \langle f(x), \phi(x)\rangle = \langle f(x), \frac{1}{\lambda^2}\phi''(x)\rangle = \frac{1}{\lambda^2}\langle f(x), \phi''(x)\rangle = \frac{1}{\lambda^2}(-1)^2\langle f''_{gen}(x), \phi(x)\rangle$$

$$= \frac{1}{\lambda^2}\langle \delta'(x) - \delta(x) + \delta(x-1), \phi(x)\rangle = \frac{\langle \delta'(x), \phi(x)\rangle - \langle \delta(x), \phi(x)\rangle + \langle \delta(x-1), \phi(x)\rangle}{\lambda^2} =$$

$$= \frac{(-1)\langle \delta(x), \phi'(x)\rangle - \phi(0) + \phi(1)}{\lambda^2} = \frac{-\phi'(0) - \phi(0) + \phi(1)}{\lambda^2} = \frac{-\lambda e^{\lambda\cdot 0} - e^{\lambda\cdot 0} + e^{\lambda\cdot 1}}{\lambda^2} =$$

$$= \frac{-\lambda - 1 + e^{\lambda}}{\lambda^2} = -\frac{1}{\lambda} - \frac{1}{\lambda^2} + \frac{e^{\lambda}}{\lambda^2} \qquad \Rightarrow \qquad I(\lambda) = \int_0^1 (1-x)e^{\lambda x}dx = -\frac{1}{\lambda} - \frac{1}{\lambda^2} + \frac{e^{\lambda}}{\lambda^2}$$

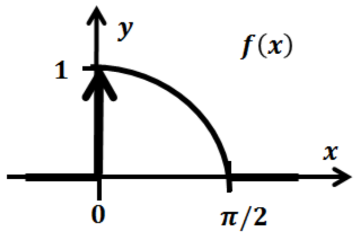
PROBLEMA 9

Encuentre sin realizar integración alguna la siguiente función explícitamente

$$\Gamma(\omega) = \int_0^{\pi/2} \cos x \cos(\omega x)\, dx$$

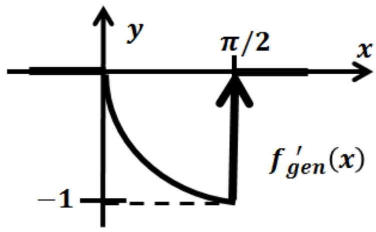
SOLUCIÓN:

Definiendo la función



$$f(x) = \begin{cases} \cos x & \text{si } \ 0 < x < \pi/2 \\ 0 & \text{si no} \end{cases}$$

Calculando la primera derivada generalizada



$$f'_{gen}(x) = \begin{cases} -\operatorname{sen} x & \text{si } \ 0 < x < \pi/2 \\ 0 & \text{si no} \end{cases} + \delta(x)$$

Calculando la segunda derivada generalizada:

$$f''_{gen}(x) = \begin{cases} -\cos x & \text{si } \ 0 < x < \pi/2 \\ 0 & \text{si no} \end{cases} + \delta'(x) + \delta(x) = -f(x) + \delta'(x) + \delta\left(x - \frac{\pi}{2}\right)$$

$$\Gamma(\omega) = \int_0^{\pi/2} \cos x \cos(\omega x)\, dx = \int_{-\infty}^{\infty} f(x) \underbrace{\cos(\omega x)}_{\phi(x)}\, dx = \int_{-\infty}^{\infty} f(x)\, \phi(x) dx = \langle f(x), \phi(x) \rangle$$

Como $\phi(x) = \cos(\omega x) \in \mathcal{C}^{\infty}(\mathbb{R})$ es una función prueba que satisface la ecuación diferencial

$$\phi''(x) + \omega^2 \phi(x) = 0 \quad \Rightarrow \quad \phi(x) = -\frac{1}{\omega^2}\phi''(x)$$

Sustituyendo esta expresión en el producto interno

$$\Gamma(\omega) = \langle f(x), \phi(x) \rangle = \langle f(x), -\frac{1}{\omega^2}\phi''(x) \rangle = -\frac{1}{\omega^2}\langle f(x), \phi''(x) \rangle = -\frac{1}{\omega^2}(-1)^2 \langle f''_{gen}(x), \phi(x) \rangle =$$

$$= -\frac{1}{\omega^2}\langle -f(x) + \delta'(x) + \delta\left(x - \frac{\pi}{2}\right), \phi(x) \rangle = \frac{\langle f(x), \phi(x) \rangle - \langle \delta'(x), \phi(x) \rangle - \langle \delta\left(x - \frac{\pi}{2}\right), \phi(x) \rangle}{\omega^2} =$$

$$= \frac{\langle f(x), \phi(x) \rangle - (-1)\langle \delta(x), \phi'(x) \rangle - \phi(\pi/2)}{\omega^2} = \frac{\langle f(x), \phi(x) \rangle + \phi'(0) - \phi(\pi/2)}{\omega^2} =$$

$$= \frac{\langle f(x), \phi(x) \rangle - \omega \sin(\omega \cdot 0) - \cos\left(\frac{\pi}{2}\omega\right)}{\omega^2} = \frac{\langle f(x), \phi(x) \rangle - \cos\left(\frac{\pi}{2}\omega\right)}{\omega^2}$$

Despejando el producto interno de la expresión anterior

$$\omega^2 \langle f(x), \phi(x) \rangle = \langle f(x), \phi(x) \rangle - \cos\left(\frac{\pi}{2}\omega\right) \quad \Rightarrow \quad (\omega^2 - 1)\langle f(x), \phi(x) \rangle = -\cos\left(\frac{\pi}{2}\omega\right)$$

$$\Rightarrow \quad \Gamma(\omega) = \langle f(x), \phi(x) \rangle = \frac{\cos\left(\frac{\pi}{2}\omega\right)}{1 - \omega^2}$$

PROBLEMA 10

Sea $f: [0,2] \to \mathbb{R}$ la función definida por

$$f(x) = \begin{cases} x & \text{si} \quad 0 \le x < 1 \\ 2 - x & \text{si} \quad 1 \le x \le 2 \end{cases}$$

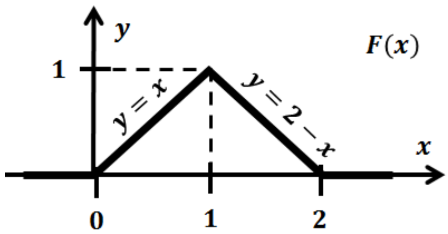
Hallar la función $g(\omega)$ definida por

$$g(\omega) = \frac{2}{\pi} \int_0^2 f(x) \operatorname{sen}(\omega x)\, dx$$

SOLUCIÓN:

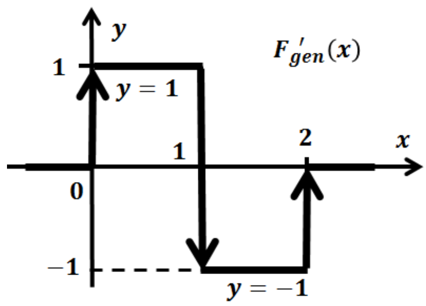
Se redefine la función $f(x)$ para todos los números reales como $F: \mathbb{R} \to \mathbb{R}$

$$F(x) = \begin{cases} x & \text{si} \ 0 \le x < 1 \\ 2 - x & \text{si} \quad 1 \le x \le 2 \\ 0 & \text{si no} \end{cases}$$



La primera derivada generalizada de la función $F(x)$ es:

$$F'_{gen}(x) = F'(x) = \begin{cases} 1 & \text{si} \quad 0 \le x < 1 \\ -1 & \text{si} \quad 1 \le x \le 2 \\ 0 & \text{si no} \end{cases}$$



Y la segunda derivada generalizada:

$$F''_{gen}(x) = \delta(x) - 2\delta(x-1) + \delta(x-2)$$

Por lo tanto

$$g(\omega) = \frac{2}{\pi} \int_0^2 f(x) \operatorname{sen}(\omega x)\, dx = \frac{2}{\pi} \int_{-\infty}^{\infty} F(x) \underbrace{\operatorname{sen}(\omega x)}_{\phi(x)}\, dx = \frac{2}{\pi} \langle F(x), \phi(x) \rangle$$

Como $\phi(x) = \operatorname{sen}(\omega x) \in C^\infty(\mathbb{R})$ es una función prueba que satisface la ecuación diferencial

$$\phi''(x) + \omega^2 \phi(x) = 0 \quad \Rightarrow \quad \phi(x) = -\frac{1}{\omega^2} \phi''(x)$$

Sustituyendo esta expresión en el producto interno

$$g(\omega) = \frac{2}{\pi} \langle F(x), \phi(x) \rangle = \frac{2}{\pi} \langle F(x), -\frac{1}{\omega^2} \phi''(x) \rangle = -\frac{2}{\pi \omega^2} \langle F(x), \phi''(x) \rangle =$$

$$= -\frac{2}{\pi\omega^2}(-1)^2\langle F''_{gen}(x), \phi(x)\rangle = -\frac{2}{\pi\omega^2}\langle F''_{gen}(x), \phi(x)\rangle =$$

$$= -\frac{2}{\pi\omega^2}\langle \delta(x) - 2\delta(x-1) + \delta(x-2), \phi(x)\rangle =$$

$$= -\frac{2}{\pi\omega^2}\langle \delta(x) - 2\delta(x-1) + \delta(x-2), \phi(x)\rangle =$$

$$= -\frac{2}{\pi\omega^2}\langle \delta(x), \phi(x)\rangle + \frac{4}{\pi\omega^2}\langle \delta(x-1), \phi(x)\rangle - \frac{2}{\pi\omega^2}\langle \delta(x-2), \phi(x)\rangle =$$

$$= -\frac{2}{\pi\omega^2}\phi(0) + \frac{4}{\pi\omega^2}\phi(1) - \frac{2}{\pi\omega^2}\phi(2)$$

Evaluando cada uno de los términos

$$\phi(x) = \operatorname{sen}(\omega x)$$

$$\phi(0) = \operatorname{sen}(0) = 0$$

$$\phi(1) = \operatorname{sen}(\omega)$$

$$\phi(2) = \operatorname{sen}(2\omega)$$

$$g(\omega) = \frac{2}{\pi}\langle F(x), \phi(x)\rangle = -\frac{2}{\pi\omega^2}\phi(0) + \frac{4}{\pi\omega^2}\phi(1) - \frac{2}{\pi\omega^2}\phi(2) =$$

$$0 + \frac{4}{\pi\omega^2}\operatorname{sen}(\omega) - \frac{2}{\pi\omega^2}\operatorname{sen}(2\omega) = \frac{2}{\pi\omega^2}(2\operatorname{sen}\omega - \operatorname{sen}(2\omega))$$

PROBLEMA 11

Para $n = 1,2,3,4 \ldots$ halle la sucesión ζ_n si esta está definida por

$$\zeta_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

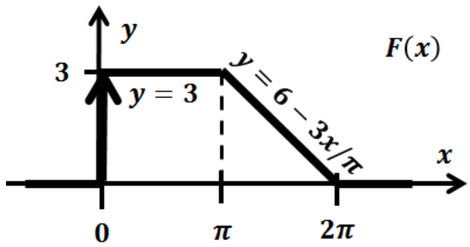
Donde $f: [0,2\pi] \to \mathbb{R}$ está definida por

$$f(x) = \begin{cases} 3 & \text{si } 0 \le x < \pi \\ 6 - \dfrac{3x}{\pi} & \text{si } \pi \le x \le 2\pi \end{cases}$$

SOLUCIÓN:

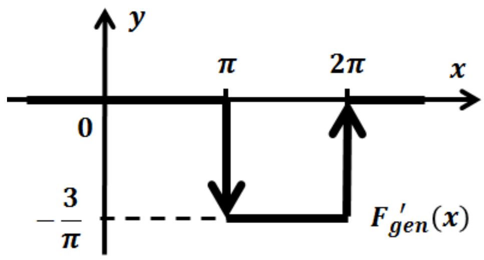
Se redefine la función $f: [0,2\pi] \to \mathbb{R}$ como $F: \mathbb{R} \to \mathbb{R}$

$$F(x) = \begin{cases} 3 & \text{si } 0 \le x < \pi \\ 6 - \dfrac{3x}{\pi} & \text{si } \pi \le x \le 2\pi \\ 0 & \text{si no} \end{cases}$$



La primera derivada generalizada de la función $F(x)$ es:

$$F'_{gen}(x) = 3\delta(x) + \begin{cases} 0 & \text{si } 0 \le x < \pi \\ -\dfrac{3}{\pi} & \text{si } \pi \le x \le 2\pi \\ 0 & \text{si no} \end{cases}$$



Y la segunda derivada generalizada:

$$F''_{gen}(x) = 3\delta'(x) - \frac{3}{\pi}\delta(x-\pi) + \frac{3}{\pi}\delta(x-2\pi)$$

$$\zeta_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} F(x)\underbrace{\cos(nx)}_{\phi(x)}\,dx = \frac{1}{\pi}\langle F(x), \phi(x)\rangle$$

Como $\phi(x) = \cos(nx) \in C^{\infty}(\mathbb{R})$ es una función prueba que satisface la ecuación diferencial

$$\phi''(x) + n^2\phi(x) = 0 \Rightarrow \phi(x) = -\frac{1}{n^2}\phi''(x)$$

Sustituyendo esta expresión en el producto interno

$$\zeta_n = \frac{1}{\pi}\langle F(x), \phi(x)\rangle = \frac{1}{\pi}\langle F(x), -\frac{1}{n^2}\phi''(x)\rangle = -\frac{1}{\pi n^2}\langle F(x), \phi''(x)\rangle =$$

$$= -\frac{1}{\pi n^2}(-1)^2\langle F''_{gen}(x), \phi(x)\rangle = -\frac{1}{\pi n^2}\langle F''_{gen}(x), \phi(x)\rangle =$$

$$= -\frac{1}{\pi n^2}\langle 3\delta'(x) - \frac{3}{\pi}\delta(x-\pi) + \frac{3}{\pi}\delta(x-2\pi), \phi(x)\rangle =$$

$$= -\frac{1}{\pi n^2}\langle 3\delta'(x), \phi(x)\rangle - \frac{1}{\pi n^2}\langle -\frac{3}{\pi}\delta(x-\pi), \phi(x)\rangle - \frac{1}{\pi n^2}\langle \frac{3}{\pi}\delta(x-2\pi), \phi(x)\rangle =$$

$$= -\frac{3(-1)}{\pi n^2}\langle \delta(x), \phi'(x)\rangle + \frac{3}{\pi^2 n^2}\langle \delta(x-\pi), \phi(x)\rangle - \frac{3}{\pi^2 n^2}\langle \delta(x-2\pi), \phi(x)\rangle =$$

$$= \frac{3}{\pi n^2}\phi'(0) + \frac{3}{\pi^2 n^2}\phi(\pi) - \frac{3}{\pi^2 n^2}\phi(2\pi)$$

Evaluando cada uno de los términos

$$\phi(x) = \cos(nx)$$

$$\phi'(x) = -n\,\mathrm{sen}(nx) \quad \Rightarrow \quad \phi'(0) = -n\,\mathrm{sen}(0) = 0$$

$$\phi(\pi) = \cos(n\pi) = (-1)^n$$

$$\phi(2\pi) = \cos(2\pi n) = 1$$

Así

$$\zeta_n = \frac{1}{\pi}\langle F(x), \phi(x)\rangle = \frac{3}{\pi n^2}\phi'(0) + \frac{3}{\pi^2 n^2}\phi(\pi) - \frac{3}{\pi^2 n^2}\phi(2\pi)$$

$$= \frac{3}{\pi n^2}\cdot 0 + \frac{3}{\pi^2 n^2}(-1)^n - \frac{3}{\pi^2 n^2}(1) = \frac{3}{\pi^2 n^2}(-1)^n - \frac{3}{\pi^2 n^2} = \frac{3}{\pi^2 n^2}((-1)^n - 1)$$

$$(-1)^n - 1 = \begin{cases} 0 & \text{si } n \text{ es par} \\ -2 & \text{si } n \text{ es impar} \end{cases}$$

$$\zeta_n = \frac{3}{\pi^2 n^2}((-1)^n - 1) = \frac{3(-2)}{\pi^2(2n-1)^2} = -\frac{6}{\pi^2(2n-1)^2} \quad , \quad n = 1,2,3,4 \ldots$$

$$\zeta_n = -\frac{6}{\pi^2(2n-1)^2} \quad , \quad n = 1,2,3,4 \ldots$$

PROBLEMA 12

Sea $f : \mathbb{R} \to \mathbb{R}$ la función definida por

$$f(x) = (x^2 - \pi^2)1_{(-\pi,\pi)}(x)$$

donde

$$1_{(-\pi,\pi)}(x) = \begin{cases} 1 & \text{si} \quad |x| < \pi \\ 0 & \text{si} \quad |x| \geq \pi \end{cases}$$

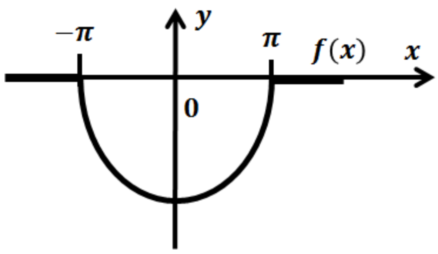
(a) Calcule $f'''_{gen}(x)$

(b) Calcule la siguiente integral para $n \in \mathbb{Z}$

$$C_n = \int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

SOLUCIÓN:

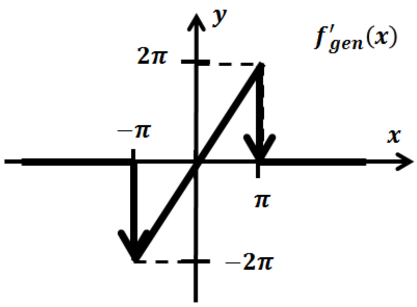
$$f(x) = (x^2 - \pi^2)1_{(-\pi,\pi)}(x) = \begin{cases} x^2 - \pi^2 & \text{si} \quad -\pi < x < \pi \\ 0 & \text{si no} \end{cases}$$



Como f es continua entonces $f'_{gen}(x) = f'(x)$

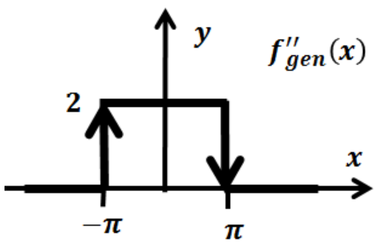
Se calcula la primera derivada generalizada

$$f'_{gen}(x) = f'(x) = \begin{cases} 2x & \text{si} \quad -\pi < x < \pi \\ 0 & \text{si no} \end{cases}$$



Luego se calcula la segunda derivada generalizada

$$f''_{gen}(x) = \begin{cases} 2x & \text{si} \quad -\pi < x < \pi \\ 0 & \text{si no} \end{cases} - 2\pi\delta(x + \pi) - 2\pi\delta(x - \pi)$$



Finalmente se halla la tercera derivada generalizada

$$f'''_{gen}(x) = 2\delta(x + \pi) - 2\delta(x - \pi) - 2\pi\delta'(x + \pi) - 2\pi\delta'(x - \pi)$$

$$C_n = \int_{-\pi}^{\pi} f(x)\underbrace{e^{-inx}}_{\phi(x)}dx = \int_{-\infty}^{\infty} f(x)\,\phi(x)dx = \langle f(x), \phi(x)\rangle$$

Sea $\phi(x) = e^{-inx} \in C^{\infty}(\mathbb{R})$ es una función prueba que satisface la ecuación diferencial

$$\phi''(x) + n^2\phi(x) = 0 \qquad \Rightarrow \qquad \phi'''(x) + n^2\phi'(x) = 0$$

$$\phi'(x) = -ine^{-inx} = -in\phi(x)$$

Entonces

$$\phi'''(x) + n^2\phi'(x) = 0 \qquad \Rightarrow \qquad \phi'''(x) + n^2\big(-in\phi(x)\big) = 0 \quad \Rightarrow \quad \phi(x) = \frac{\phi'''(x)}{in^3}$$

$$C_n = \langle f(x), \phi(x)\rangle = \langle f(x), \frac{\phi'''(x)}{in^3}\rangle = \frac{1}{in^3}(-1)^3\langle f_{gen}'''(x), \phi(x)\rangle =$$

$$= -\frac{1}{in^3}\langle 2\delta(x+\pi) - 2\delta(x-\pi) - 2\pi\delta'(x+\pi) - 2\pi\delta'(x-\pi), \phi(x)\rangle =$$

$$= -\frac{2}{in^3}\langle \delta(x+\pi), \phi(x)\rangle + \frac{2}{in^3}\langle \delta(x-\pi), \phi(x)\rangle + \frac{2\pi}{in^3}\langle \delta'(x+\pi), \phi(x)\rangle + \frac{2\pi}{in^3}\langle \delta'(x-\pi), \phi(x)\rangle =$$

$$= -\frac{2}{in^3}\phi(-\pi) + \frac{2}{in^3}\phi(\pi) + \frac{2\pi}{in^3}(-1)^1\langle \delta(x+\pi), \phi'(x)\rangle + \frac{2\pi}{in^3}(-1)^1\langle \delta(x-\pi), \phi'(x)\rangle =$$

$$= \frac{2}{in^3}\big(\phi(\pi) - \phi(-\pi)\big) - \frac{2\pi}{in^3}\big(\phi'(-\pi) + \phi'(\pi)\big)$$

$$\phi(x) = e^{-inx} \qquad \Rightarrow \qquad \begin{cases} \phi(\pi) = e^{-in\pi} \\ \phi(-\pi) = e^{in\pi} \end{cases}$$

$$\phi'(x) = -in\phi(x) \qquad \Rightarrow \qquad \begin{cases} \phi'(\pi) = -ine^{-in\pi} \\ \phi'(-\pi) = -ine^{in\pi} \end{cases}$$

Por lo tanto

$$C_n = \langle f(x), \phi(x)\rangle = \frac{2}{in^3}\big(e^{-in\pi} - e^{in\pi}\big) - \frac{2\pi}{in^3}\big(-ine^{in\pi} - ine^{-in\pi}\big) =$$

$$-\frac{2}{in^3}\underbrace{\big(e^{in\pi} - e^{-in\pi}\big)}_{2i\,\text{sen}(n\pi)} + \frac{2\pi}{n^2}\underbrace{\big(e^{in\pi} + e^{-in\pi}\big)}_{2\cos(n\pi)} = -\frac{2}{in^3}(2i\,\text{sen}(n\pi)) + \frac{2\pi}{n^2}(2\cos(n\pi)) =$$

$$= -\frac{2}{in^3}\cdot 0 + \frac{2\pi}{n^2}(2(-1)^n) = \frac{4\pi}{n^2}(-1)^n$$

Por lo que los C_n vienen dados por

$$C_n = \frac{4\pi}{n^2}(-1)^n$$

PROBLEMA 13

Sea $\omega > 0$. Halle en forma explícita una función causal $u(t)$ tal que

$$tH(t) * u(t) = \frac{1-\cos(\omega t)}{\omega^2} H(t)$$

SOLUCIÓN:

Sea $g(t) = tH(t)$ y sea $f(t) = tH(t) * u(t) = g(t) * u(t)$

Derivando dos veces la función $f(t)$:

$$f''_{gen}(t) = g''_{gen}(t) * u(t) = \frac{d^2}{dt^2}\left(\frac{1-\cos(\omega t)}{\omega^2} H(t)\right)$$

$$g(t) = tH(t) \quad \Rightarrow \quad g'_{gen}(t) = H(t) \quad \Rightarrow \quad g''_{gen}(t) = H'(t) = \delta(t)$$

Así

$$f''_{gen}(t) = \delta(t) * u(t) = \frac{d^2}{dt^2}\left(\frac{1-\cos(\omega t)}{\omega^2} H(t)\right)$$

Se tiene que $\delta(t) * u(t) = u(t)$ ya que la distribución $\delta(t)$ es el elemento neutro de la convolución, así

$$u(t) = \frac{d^2}{dt^2}\left(\frac{1-\cos(\omega t)}{\omega^2} H(t)\right) = \frac{d^2}{dt^2}\big(H(t)y(t)\big)$$

Con $y(t) = \frac{1-\cos(\omega t)}{\omega^2}$ se desarrolla hasta la segunda derivada generalizada:

$$\frac{d^2}{dt^2}\big(H(t)y(t)\big) = \frac{d}{dt}\big(H(t)y'(t) + y(0)\delta(t)\big) = H(t)y''(t) + y(0)\delta'(t) + y'(0)\delta(t)$$

$$y(t) = \frac{1-\cos(\omega t)}{\omega^2} \quad \Rightarrow \quad y'(t) = 0 + \frac{\omega\ \operatorname{sen}(\omega t)}{\omega^2} \quad \Rightarrow \quad y''(t) = \frac{\omega^2 \cos(\omega t)}{\omega^2} = \cos(\omega t)$$

$$y(0) = \frac{1-\cos(0)}{\omega^2} = \frac{1-1}{\omega^2} = 0$$

$$y'(0) = \frac{\omega\ \operatorname{sen}(0)}{\omega^2} = 0$$

$$u(t) = \frac{d^2}{dt^2}\left(\frac{1-\cos(\omega t)}{\omega^2} H(t)\right) = H(t)y''(t) + 0\cdot\delta'(t) + 0\cdot\delta(t) = H(t)\cos(\omega t)$$

PROBLEMA 14

Hallar una distribución causal $u(t)$ cuya transformada de Laplace es

$$U(z)=\frac{z^3}{z^2+3z+2}$$

Utilizando teoremas operacionales.

SOLUCIÓN:

Dividiendo los polinomios:

z^3	z^2+3z+2
$-z^3-3z^2-2z$	$z-3$
$-3z^2-2z$	
$+3z^2+9z+6$	
$7z+6$	

$$U(z)=\frac{z^3}{z^2+3z+2}=z-3+\frac{7z+6}{z^2+3z+2}=z-3+\frac{7z+6}{(z+1)(z+2)}=z-3+\frac{A}{z+1}+\frac{B}{z+2}$$

$$\frac{7z+6}{(z+1)(z+2)}=\frac{A}{z+1}+\frac{B}{z+2}=\frac{A(z+1)+B(z+2)}{(z+1)(z+2)}$$

$$7z+6=A(z+1)+B(z+2)\quad\Rightarrow\quad\begin{cases}z=-1: & 7(-1)+6=A(-1+1)+B(-1+2)\Rightarrow B=-1\\ z=-2: & 7(-2)+6=A(-2+1)+B(-2+2)\Rightarrow A=8\end{cases}$$

$$U(z)=\frac{z^3}{z^2+3z+2}=z-3+\frac{8}{z+1}-\frac{1}{z+2}$$

Luego se recupera la función $u(t)$ mediante la transformada inversa de Laplace definida por

$$u(t)\equiv\mathcal{L}^{-1}[U(z)]=\lim_{R\to\infty}\frac{1}{2\pi i}\oint_{|z|=R}H(t)U(z)e^{zt}dz$$

Tomando transformada inversa de Laplace en la ecuación anterior

$$\mathcal{L}^{-1}[U(z)]=\mathcal{L}^{-1}[z]-3\mathcal{L}^{-1}[1]+8\mathcal{L}^{-1}\left[\frac{1}{z+1}\right]-\mathcal{L}^{-1}\left[\frac{1}{z+2}\right]$$

$$u(t)=\delta'(t)-3\delta(t)+8H(t)e^{-t}-H(t)e^{-2t}=\delta'(t)-3\delta(t)+H(t)(8e^{-t}-e^{-2t})$$

PROBLEMA 15

Sea $a > 0$. Hallar una distribución causal $u(t)$ cuya transformada de Laplace es

$$U(z) = \frac{z^2 - a^2}{(z^2 + a^2)^2}$$

Utilizando el método de los residuos.

SOLUCIÓN:

$$U(z) = \frac{z^2 - a^2}{(z^2 + a^2)^2} = \frac{z^2 - a^2}{(z - ia)^2 (z + ia)^2}$$

$$\operatorname*{Res}_{z=ia} U(z)e^{zt} = \operatorname*{Res}_{z=ia} \frac{z^2 - a^2}{(z - ia)^2 (z + ia)^2} e^{zt} = \frac{1}{1!} \lim_{z \to ia} \frac{d}{dz} \left((z - ia)^2 \frac{z^2 - a^2}{(z - ia)^2 (z + ia)^2} e^{zt} \right)$$

$$= \lim_{z \to ia} \frac{d}{dz} \left(\frac{z^2 - a^2}{(z + ia)^2} e^{zt} \right) = \lim_{z \to ia} \left(e^{zt} \frac{d}{dz} \left(\frac{z^2 - a^2}{(z + ia)^2} \right) + te^{zt} \cdot \frac{z^2 - a^2}{(z + ia)^2} \right) =$$

$$= \lim_{z \to ia} \left(e^{zt} \frac{2z(z + ia)^2 - (z^2 - a^2) \cdot 2(z + ia)}{(z + ia)^4} + te^{zt} \cdot \frac{z^2 - a^2}{(z + ia)^2} \right) =$$

$$= e^{iat} \frac{2ia(ia + ia)^2 - ((ia)^2 - a^2) \cdot 2(ia + ia)}{(ia + ia)^4} + te^{iat} \cdot \frac{(ia)^2 - a^2}{(ia + ia)^2} =$$

$$= e^{iat} \frac{(2ia)^3 - (2ia)^3}{(2ia)^4} + te^{iat} \cdot \frac{-2a^2}{(2ia)^2} = te^{iat} \cdot \frac{-2a^2}{-4a^2} = \frac{1}{2} te^{iat}$$

$$\operatorname*{Res}_{z=-ia} U(z)e^{zt} = \operatorname*{Res}_{z=-ia} \frac{z^2 - a^2}{(z - ia)^2 (z + ia)^2} e^{zt}$$

$$= \frac{1}{1!} \lim_{z \to -ia} \frac{d}{dz} \left((z + ia)^2 \frac{z^2 - a^2}{(z - ia)^2 (z + ia)^2} e^{zt} \right)$$

$$= \lim_{z \to -ia} \frac{d}{dz} \left(\frac{z^2 - a^2}{(z - ia)^2} e^{zt} \right) = \lim_{z \to -ia} \left(e^{zt} \frac{d}{dz} \left(\frac{z^2 - a^2}{(z - ia)^2} \right) + te^{zt} \cdot \frac{z^2 - a^2}{(z - ia)^2} \right) =$$

$$= \lim_{z \to -ia} \left(e^{zt} \frac{2z(z - ia)^2 - (z^2 - a^2) \cdot 2(z - ia)}{(z - ia)^4} + te^{zt} \cdot \frac{z^2 - a^2}{(z - ia)^2} \right) =$$

$$= e^{-iat} \frac{-2ia(-ia - ia)^2 - ((-ia)^2 - a^2) \cdot 2(-ia - ia)}{(-ia - ia)^4} + te^{-iat} \cdot \frac{(-ia)^2 - a^2}{(-ia - ia)^2} =$$

$$= e^{-iat} \frac{-(2ia)^3 + (2ia)^3}{(2ia)^4} + te^{-iat} \cdot \frac{-2a^2}{(2ia)^2} = te^{-iat} \cdot \frac{-2a^2}{-4a^2} = \frac{1}{2} te^{-iat}$$

$$u(t) = H(t) \left(\operatorname*{Res}_{z=ia} \frac{z^2 - a^2}{(z^2 + a^2)^2} e^{zt} + \operatorname*{Res}_{z=-ia} \frac{z^2 - a^2}{(z^2 + a^2)^2} e^{zt} \right) = H(t) \left(\frac{1}{2} te^{iat} + \frac{1}{2} te^{-iat} \right)$$
$$= tH(t) \cos t$$

PROBLEMA 16

Sea $\lambda > 0$. Halle en forma explícita una función causal $u(t)$ tal que

$$u(t) = tH(t) * H(t)e^{\lambda t}$$

SOLUCIÓN:

- **MÉTODO 1.** Usando sólo propiedades de convolución

Sea $g(t) = tH(t)$ y sea $f(t) = H(t)e^{\lambda t}$. Derivando dos veces la función $u(t)$:

$$u''_{gen}(t) = g''_{gen}(t) * f(t) = g(t) * f''_{gen}(t)$$

$$g(t) = tH(t) \quad \Rightarrow \quad g'_{gen}(t) = H(t) \quad \Rightarrow \quad g''_{gen}(t) = H'(t) = \delta(t)$$

$$f(t) = H(t)e^{\lambda t} \;\Rightarrow\; f'_{gen}(t) = \lambda H(t)e^{\lambda t} + e^0\delta(t) \;\Rightarrow\; f''_{gen}(t) = \lambda^2 H(t)e^{\lambda t} + e^0\delta'(t) + \lambda e^0\delta(t)$$

$$f''_{gen}(t) = \lambda^2 H(t)e^{\lambda t} + \delta'(t) + \lambda\delta(t) = \lambda^2 f(t) + \delta'(t) + \lambda\delta(t)$$

$$g''_{gen}(t) * f(t) = g(t) * f''_{gen}(t) \quad \Rightarrow \quad \delta(t) * f(t) = g(t) * \left(\lambda^2 f(t) + \delta'(t) + \lambda\delta(t)\right)$$

Se tiene que $\delta(t) * f(t) = f(t)$ ya que la distribución $\delta(t)$ es el elemento neutro de la convolución, luego aplicando propiedades distributiva y conmutativa de la convolución

$$f(t) = g(t) * \left(\lambda^2 f(t) + \delta'(t) + \lambda\delta(t)\right) = \lambda^2 g(t) * f(t) + g(t) * \delta'(t) + \lambda g(t) * \delta(t)$$

$$f(t) = \lambda^2 \underbrace{f(t) * g(t)}_{u(t)} + g'_{gen}(t) * \delta(t) + \lambda g(t) = \lambda^2 u(t) + g'_{gen}(t) + \lambda g(t)$$

Se despeja la función $u(t)$:

$$u(t) = \frac{f(t) - g'_{gen}(t) - \lambda g(t)}{\lambda^2} = \frac{H(t)e^{\lambda t} - H(t) - \lambda tH(t)}{\lambda^2} = H(t)\left(\frac{1}{\lambda^2}e^{\lambda t} - \frac{1}{\lambda}t - \frac{1}{\lambda^2}\right)$$

- **MÉTODO 2.** Usando transformadas de Laplace.

$$G(z) \equiv \mathcal{L}[g(t)] = \mathcal{L}[tH(t)] = (-1)\frac{d}{dz}\left(\frac{1}{z}\right) = \frac{1}{z^2} \qquad\qquad F(z) \equiv \mathcal{L}[f(t)] = \mathcal{L}\left[H(t)e^{\lambda t}\right] = \frac{1}{z-\lambda}$$

$$U(z) = \mathcal{L}[g(t) * f(t)] = G(z)F(z) = \frac{1}{z^2}\cdot\frac{1}{z-\lambda} = \frac{1}{z^2(z-\lambda)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-\lambda} = \frac{Az(z-\lambda) + B(z-\lambda) + Cz^2}{z^2(z-\lambda)}$$

$$1 = A(z^2 - \lambda z) + B(z-\lambda) + Cz^2 \;\Rightarrow \begin{cases} z^2:\; A + C = 0 \\ z^1:\, -\lambda A + B = 0 \\ z^0:\, -\lambda B = 1 \end{cases} \Rightarrow \begin{cases} z^2:\; C = -A = 1/\lambda^2 \\ z^1:\, A = B/\lambda = -1/\lambda^2 \\ z^0:\;\; B = -1/\lambda \end{cases}$$

$$U(z) = \frac{1}{z^2(z-\lambda)} = -\frac{1}{\lambda^2}\cdot\frac{1}{z} - \frac{1}{\lambda}\cdot\frac{1}{z^2} + \frac{1}{\lambda^2}\cdot\frac{1}{z-\lambda} \;\Rightarrow\; \mathcal{L}^{-1}[U(z)] = -\frac{1}{\lambda^2}\mathcal{L}^{-1}\left[\frac{1}{z}\right] - \frac{1}{\lambda}\mathcal{L}^{-1}\left[\frac{1}{z^2}\right] + \frac{1}{\lambda^2}\mathcal{L}^{-1}\left[\frac{1}{z-\lambda}\right]$$

$$u(t) = -\frac{1}{\lambda^2}H(t) - \frac{1}{\lambda}tH(t) + \frac{1}{\lambda^2}H(t)e^{\lambda t} = H(t)\left(\frac{1}{\lambda^2}e^{\lambda t} - \frac{1}{\lambda}t - \frac{1}{\lambda^2}\right)$$

PROBLEMA 17

Halle $u(t)$ si

$$u(t) = (3t-2)H(t) * H(t)\,\text{sen}\,t$$

SOLUCIÓN:

- **MÉTODO 1.** Usando sólo propiedades de convolución.

$$u(t) = \underbrace{(3t-2)H(t)}_{f(t)} * \underbrace{H(t)\,\text{sen}\,t}_{g(t)} = f(t) * g(t)$$

$$u''_{gen}(t) = f''_{gen}(t) * g(t) = f(t) * g''_{gen}(t)$$

$$f(t) = (3t-2)H(t) = 3tH(t) - 2H(t) \quad \Rightarrow \quad f'_{gen}(t) = -2\delta(t) + 3H(t)$$

$$f''_{gen}(t) = \frac{d}{dt}\big(-2\delta(t) + 3H(t)\big) = -2\delta'(t) + 3\delta(t)$$

Por lo tanto

$$u''_{gen}(t) = f''_{gen}(t) * g(t) = \big(-2\delta'(t) + 3\delta(t)\big) * H(t)\,\text{sen}\,t =$$

$$-2\delta'(t) * H(t)\,\text{sen}\,t + 3\delta(t) * H(t)\,\text{sen}\,t = -2\delta(t) * \frac{d}{dt}(H(t)\,\text{sen}\,t) + 3H(t)\,\text{sen}\,t$$
$$= -2\delta(t) * H(t)\cos t + 3H(t)\,\text{sen}\,t = -2H(t)\cos t + 3H(t)\,\text{sen}\,t =$$

$$= H(t)(3\,\text{sen}\,t - 2\cos t)$$

Luego, se recupera $u(t)$ integrando dos veces la expresión anterior:

$$u''_{gen}(t) = H(t)(3\,\text{sen}\,t - 2\cos t)$$

$$u'_{gen}(t) = \int_{-\infty}^{t} H(t)(3\,\text{sen}\,t - 2\cos t)\,dt = H(t)\int_{0}^{t}(3\,\text{sen}\,t - 2\cos t)\,dt =$$

$$H(t)(-3\cos t - 2\,\text{sen}\,t) - H(t)(-3\cos(0) - 2\,\text{sen}(0))$$
$$= H(t)(-3\cos t - 2\,\text{sen}\,t + 3)$$

$$u(t) = \int_{-\infty}^{t} H(t)(-3\cos t - 2\,\text{sen}\,t + 3)\,dt = H(t)\int_{0}^{t}(-3\cos t - 2\,\text{sen}\,t + 3)\,dt =$$

$$H(t)(-3\,\text{sen}\,t + 2\cos t + 3t) - H(t)(-3\,\text{sen}(0) + 2\cos(0) + 3\cdot 0)$$
$$= H(t)(-3\,\text{sen}\,t + 2\cos t + 3t - 2)$$

$$u(t) = (3t-2)H(t) * H(t)\,\text{sen}\,t = H(t)(-3\,\text{sen}\,t + 2\cos t + 3t - 2)$$

- **MÉTODO 2.** Usando transformadas de Laplace.

$$u(t) = \underbrace{(3t-2)H(t)}_{f(t)} * \underbrace{H(t)\,\mathrm{sen}\,t}_{g(t)} = f(t) * g(t)$$

$$u(t) = f(t) * g(t) \quad \Rightarrow \quad U(z) \equiv \mathcal{L}[u(t)] = \mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] = F(z)G(z)$$

$$f(t) = (3t-2)H(t) \quad \Rightarrow \quad F(z) \equiv \mathcal{L}[f(t)] = \mathcal{L}[(3t-2)H(t)] = 3\mathcal{L}[tH(t)] - 2\mathcal{L}[H(t)]$$

$$g(t) = H(t)\,\mathrm{sen}\,t \quad \Rightarrow \quad G(z) \equiv \mathcal{L}[g(t)] = \mathcal{L}[H(t)\,\mathrm{sen}\,t]$$

Calculando la transformada de Laplace de cada uno de los términos por separado

$$\mathcal{L}[H(t)] = \frac{1}{z} \qquad \mathcal{L}[tH(t)] = (-1)\frac{d}{dz}\left(\frac{1}{z}\right) = \frac{1}{z^2} \qquad \mathcal{L}[H(t)\,\mathrm{sen}\,t] = \frac{1}{z^2+1}$$

Sustituyendo estas expresiones en las funciones $F(z)$ y $G(z)$:

$$F(z) = 3\mathcal{L}[tH(t)] - 2\mathcal{L}[H(t)] = \frac{3}{z^2} - \frac{2}{z}$$

$$G(z) = \mathcal{L}[H(t)\,\mathrm{sen}\,t] = \frac{1}{z^2+1}$$

$$U(z) = F(z)G(z) = \left(\frac{3}{z^2} - \frac{2}{z}\right)\cdot\frac{1}{z^2+1} = \frac{3-2z}{z^2}\cdot\frac{1}{z^2+1} = \frac{3-2z}{z^2(z^2+1)}$$

Separando $U(z)$ en fracciones simples:

$$U(z) = \frac{3-2z}{z^2(z^2+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{Cz+D}{z^2+1} = \frac{Az(z^2+1) + B(z^2+1) + z^2(Cz+D)}{z^2(z^2+1)}$$

$$3 - 2z = A(z^3+z) + B(z^2+1) + (Cz^3 + Dz^2) \quad \Rightarrow \begin{cases} z^3: \quad A + C = 0 \Rightarrow C = -A = 2 \\ z^2: \quad B + D = 0 \Rightarrow D = -B = -3 \\ z^1: \quad A = -2 \\ z^0: \quad B = 3 \end{cases}$$

$$U(z) = \frac{3-2z}{z^2(z^2+1)} = -\frac{2}{z} + \frac{3}{z^2} + \frac{2z-3}{z^2+1} = -\frac{2}{z} + \frac{3}{z^2} + 2\frac{z}{z^2+1} - 3\frac{1}{z^2+1}$$

Luego se recupera la función $u(t)$ mediante la transformada inversa de Laplace definida por

$$u(t) \equiv \mathcal{L}^{-1}[U(z)] = \lim_{R\to\infty}\frac{1}{2\pi i}\oint_{|z|=R} H(t)U(z)e^{zt}dz$$

Tomando transformada inversa de Laplace en la ecuación anterior

$$\mathcal{L}^{-1}[U(z)] = -2\mathcal{L}^{-1}\left[\frac{1}{z}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{z^2}\right] + 2\mathcal{L}^{-1}\left[\frac{z}{z^2+1}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{z^2+1}\right]$$

$$u(t) = -2H(t) + 3tH(t) + 2H(t)\cos t - 3H(t)\,\mathrm{sen}\,t = H(t)(2\cos t - 3\,\mathrm{sen}\,t - 2 + 3t)$$

PROBLEMA 18

Sean $\alpha\,,\beta > 0$. Halle en forma explícita una función causal $u(t)$ tal que

$$H(t)\,\mathrm{sen}(\alpha t) * u(t) = H(t)\frac{\alpha\,\mathrm{senh}(\beta t) - \beta\,\mathrm{sen}(\alpha t)}{\alpha^2+\beta^2}$$

SOLUCIÓN:

$$\underbrace{H(t)\,\mathrm{sen}(\alpha t)}_{g(t)} * u(t) = \underbrace{H(t)\frac{\alpha\,\mathrm{senh}(\beta t) - \beta\,\mathrm{sen}(\alpha t)}{\alpha^2+\beta^2}}_{f(t)}$$

$$g(t) * u(t) = f(t) \quad \Rightarrow \quad F(z) \equiv \mathcal{L}[f(t)] = \mathcal{L}[g(t) * u(t)] = \mathcal{L}[g(t)]\mathcal{L}[u(t)] = G(z)U(z)$$

$$G(z)U(z) = F(z) \quad \Rightarrow \quad U(z) = \frac{F(z)}{G(z)}$$

$$F(z) = \mathcal{L}\left[H(t)\frac{\alpha\,\mathrm{senh}(\beta t) - \beta\,\mathrm{sen}(\alpha t)}{\alpha^2+\beta^2}\right] = \frac{\alpha}{\alpha^2+\beta^2}\mathcal{L}[H(t)\,\mathrm{senh}(\beta t)] - \frac{\beta}{\alpha^2+\beta^2}\mathcal{L}[H(t)\,\mathrm{sen}(\alpha t)]$$

$$G(z) \equiv \mathcal{L}[g(t)] = \mathcal{L}[H(t)\,\mathrm{sen}(\alpha t)]$$

Calculando la transformada de Laplace de cada uno de los términos por separado

$$\mathcal{L}[H(t)\,\mathrm{senh}(\beta t)] = \frac{\beta}{z^2-\beta^2} \qquad \mathcal{L}[H(t)\,\mathrm{sen}(\alpha t)] = \frac{\alpha}{z^2+\alpha^2}$$

Sustituyendo estas expresiones en las funciones $F(z)$ y $G(z)$:

$$F(z) = \frac{\alpha}{\alpha^2+\beta^2}\cdot\frac{\beta}{z^2-\beta^2} - \frac{\beta}{\alpha^2+\beta^2}\cdot\frac{\alpha}{z^2+\alpha^2} = \frac{\alpha\beta}{\alpha^2+\beta^2}\left(\frac{1}{z^2-\beta^2} - \frac{1}{z^2+\alpha^2}\right)$$

$$= \frac{\alpha\beta}{\alpha^2+\beta^2}\left(\frac{z^2+\alpha^2-z^2+\beta^2}{(z^2-\beta^2)(z^2+\alpha^2)}\right) = \frac{\alpha\beta}{(z^2-\beta^2)(z^2+\alpha^2)}$$

$$G(z) = \mathcal{L}[H(t)\,\mathrm{sen}(\alpha t)] = \frac{\alpha}{z^2+\alpha^2}$$

$$U(z) = \frac{F(z)}{G(z)} = \frac{\dfrac{\alpha\beta}{(z^2-\beta^2)(z^2+\alpha^2)}}{\dfrac{\alpha}{z^2+\alpha^2}} = \frac{\beta}{z^2-\beta^2}$$

Luego se recupera la función $u(t)$ mediante la transformada inversa de Laplace definida por

$$u(t) \equiv \mathcal{L}^{-1}[U(z)] = \lim_{R\to\infty}\frac{1}{2\pi i}\oint_{|z|=R} H(t)U(z)e^{zt}dz$$

Tomando transformada inversa de Laplace en la ecuación anterior

$$\mathcal{L}^{-1}[U(z)] = \mathcal{L}^{-1}\left[\frac{\beta}{z^2-\beta^2}\right] \quad \Rightarrow \quad u(t) = H(t)\,\mathrm{senh}(\beta t)$$

PROBLEMA 19

Sea $u(t)$ una función causal tal que

$$H(t)\cos t * u(t) = H(t-1)e^t$$

Determine $u(t)$ explícitamente.

SOLUCIÓN:

$$\underbrace{H(t)\cos t}_{g(t)} * u(t) = \underbrace{H(t-1)e^t}_{f(t)}$$

$$g(t) = H(t)\cos t \qquad \Rightarrow \qquad G(z) \equiv \mathcal{L}[g(t)] = \mathcal{L}[H(t)\cos t] = \frac{z}{z^2+1}$$

$$f(t) = H(t-1)e^t \qquad \Rightarrow \qquad F(z) \equiv \mathcal{L}[f(t)] = \mathcal{L}[H(t-1)e^t] = \frac{1}{z}e^{-z}\Bigg|_{z-1} = \frac{1}{z-1}e^{-(z-1)}$$

$$g(t) * u(t) = f(t) \qquad \Rightarrow \qquad \mathcal{L}[g(t) * u(t)] = \mathcal{L}[f(t)] \qquad \Rightarrow \qquad G(z)U(z) = F(z)$$

$$U(z) = \frac{F(z)}{G(z)} = \frac{\frac{1}{z-1}e^{-(z-1)}}{\frac{z}{z^2+1}} = \frac{z^2+1}{z(z-1)}e^{-(z-1)}$$

Se divide el polinomio y luego se separa en fracciones simples

$$\begin{array}{r|l} z^2+1 & z^2-z \\ \underline{-z^2+z} & 1+\dfrac{z+1}{z^2-z} \\ z+1 & \end{array} \qquad \Rightarrow \qquad \frac{z^2+1}{z(z-1)} = 1 + \frac{z+1}{z(z-1)}$$

$$\frac{z+1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{A(z-1)+Bz}{z(z-1)} \qquad \Rightarrow \qquad z+1 = A(z-1)+Bz$$

$$z+1 = A(z-1)+Bz \qquad \Rightarrow \qquad \begin{cases} z=0: \ \ 0+1 = A(0-1) \quad \Rightarrow \quad A=-1 \\ z=1: \ \ 1+1 = B(1) \quad \Rightarrow \quad B=2 \end{cases}$$

$$U(z) = \frac{z^2+1}{z(z-1)}e^{-(z-1)} = \left(1 - \frac{1}{z} + \frac{2}{z-1}\right)e^{-(z-1)} = e\left(1 - \frac{1}{z} + \frac{2}{z-1}\right)e^{-z}$$

Se toma transformada inversa de Laplace

$$\mathcal{L}^{-1}[U(z)] = u(t) \qquad \mathcal{L}^{-1}\left[\frac{1}{z}\right] = H(t) \qquad \mathcal{L}^{-1}[e^{-z}] = \delta(t-1)$$

$$\mathcal{L}^{-1}[1] = \delta(t) \qquad \mathcal{L}^{-1}\left[\frac{1}{z-1}\right] = H(t)e^t$$

$$u(t) = e\mathcal{L}^{-1}\left[1 - \frac{1}{z} + \frac{2}{z-1}\right] * \mathcal{L}^{-1}[e^{-z}] = e(\delta(t) - H(t) + 2H(t)e^t) * \delta(t-1)$$

$$u(t) = e(\delta(t) - H(t) + 2H(t)e^t)\Bigg|_{t-1} = e(\delta(t-1) - H(t-1) + 2H(t-1)e^{t-1})$$

PROBLEMA 20

Sea $\omega > 0$ y la función f dada por:

$$f(t) = \left(H(t) - H\left(t - \frac{\pi}{\omega}\right)\right)\operatorname{sen}(\omega t)$$

(a) Calcule la transformada de Laplace $F(z)$ de la función f .

(b) Se desea resolver la ecuación

$$u''_{gen}(t) + \omega^2 u(t) = \delta(t) + \delta\left(t - \frac{\pi}{\omega}\right)$$

Calcule la transformada de Laplace $U(z)$ de la función $u(t)$ y relacione $f(t)$ con $u(t)$.

SOLUCIÓN:

(a) Conociendo que $\operatorname{sen}\theta = -\operatorname{sen}(\theta - \pi)$

$$f(t) = \left(H(t) - H\left(t - \frac{\pi}{\omega}\right)\right)\operatorname{sen}(\omega t) = H(t)\operatorname{sen}(\omega t) - H\left(t - \frac{\pi}{\omega}\right)\operatorname{sen}(\omega t) =$$

$$= H(t)\operatorname{sen}(\omega t) - H\left(t - \frac{\pi}{\omega}\right)(-\operatorname{sen}(\omega t - \pi)) = H(t)\operatorname{sen}(\omega t) + H\left(t - \frac{\pi}{\omega}\right)\operatorname{sen}\left(\omega\left(t - \frac{\pi}{\omega}\right)\right)$$

Sea $F: \mathbb{C} \to \mathbb{C}$ la transformada de Laplace de $u(t)$ definida por

$$F(z) \equiv \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-zt}dt$$

$$\mathcal{L}[H(t)\operatorname{sen}(\omega t)] = \frac{\omega}{z^2 + \omega^2}$$

$$\mathcal{L}\left[H\left(t - \frac{\pi}{\omega}\right)\operatorname{sen}\left(\omega\left(t - \frac{\pi}{\omega}\right)\right)\right] = \mathcal{L}\left[\delta\left(t - \frac{\pi}{\omega}\right) * H(t)\operatorname{sen}(\omega t)\right] = e^{-\frac{\pi}{\omega}z}\mathcal{L}[H(t)\operatorname{sen}(\omega t)] = \frac{\omega}{z^2 + \omega^2}e^{-\frac{\pi}{\omega}z}$$

$$F(z) = \mathcal{L}[H(t)\operatorname{sen}(\omega t)] + \mathcal{L}\left[H\left(t - \frac{\pi}{\omega}\right)\operatorname{sen}\left(\omega\left(t - \frac{\pi}{\omega}\right)\right)\right] = \frac{\omega}{z^2 + \omega^2} + \frac{\omega}{z^2 + \omega^2}e^{-\frac{\pi}{\omega}z} = \frac{\omega}{z^2 + \omega^2}\left(1 + e^{-\frac{\pi}{\omega}z}\right)$$

(b) Se toman transformadas de Laplace a la ecuación de derivadas generalizadas:

$$u''_{gen}(t) + \omega^2 u(t) = \delta(t) + \delta\left(t - \frac{\pi}{\omega}\right) \quad \Rightarrow \quad \mathcal{L}[u''_{gen}(t)] + \omega^2\mathcal{L}[u(t)] = \mathcal{L}[\delta(t)] + \mathcal{L}\left[\delta\left(t - \frac{\pi}{\omega}\right)\right]$$

$$\mathcal{L}[u(t)] = U(z) \qquad \mathcal{L}[u''_{gen}(t)] = z^2 U(z)$$

$$\mathcal{L}[\delta(t)] = 1 \qquad \mathcal{L}\left[\delta\left(t - \frac{\pi}{\omega}\right)\right] = e^{-\frac{\pi}{\omega}z}\mathcal{L}[\delta(t)] = e^{-\frac{\pi}{\omega}z}$$

Se sustituyen cada uno de los términos en la expresión anterior

$$\mathcal{L}[u''_{gen}(t)] + \omega^2\mathcal{L}[u(t)] = \mathcal{L}[\delta(t)] + \mathcal{L}\left[\delta\left(t - \frac{\pi}{\omega}\right)\right] \quad \Rightarrow \quad z^2 U(z) + \omega^2 U(z) = 1 + e^{-\frac{\pi}{\omega}z}$$

$$U(z) = \frac{1}{z^2 + \omega^2}\left(1 + e^{-\frac{\pi}{\omega}z}\right) = \frac{1}{\omega}F(z) \quad \Rightarrow \quad u(t) = \mathcal{L}^{-1}[U(z)] = \frac{1}{\omega}\mathcal{L}^{-1}[F(z)] = \frac{f(t)}{\omega}$$

PROBLEMA 21

Resuelva el siguiente problema de valores iniciales

$$\begin{cases} t\, u''_{gen}(t) + u'_{gen}(t) = \delta'(t-\pi) \\ \\ u(0) = 1 \end{cases}$$

donde $u(t)$ es una función causal.

SOLUCIÓN:

Se toman transformadas de Laplace en la ecuación de derivadas generalizadas, definiendo la transformada de Laplace de $u(t)$ como

$$U(z) \equiv \mathcal{L}[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-zt} dt$$

$$\mathcal{L}\left[u'_{gen}(t)\right] = zU(z)$$

$$\mathcal{L}\left[t\, u''_{gen}(t)\right] = -\frac{d}{dz}\left(z^2 U(z)\right) = -\left(2zU(z) + z^2 U'(z)\right)$$

$$\mathcal{L}[\delta'(t-\pi)] = ze^{-\pi z}$$

Sustituyendo las expresiones:

$$\mathcal{L}\left[t\, u''_{gen}(t)\right] + \mathcal{L}\left[u'_{gen}(t)\right] = \mathcal{L}[\delta'(t-\pi)] \Rightarrow -2zU(z) - z^2 U'(z) + zU(z) = ze^{-\pi z}$$

$$-zU(z) - z^2 U'(z) = ze^{-\pi z} \Rightarrow -z\left(U(z) + zU'(z)\right) = ze^{-\pi z}$$

$$U(z) + zU'(z) = -e^{-\pi z} \Rightarrow \frac{d}{dz}\left(zU(z)\right) = -e^{-\pi z} \Rightarrow zU(z) = -\int e^{-\pi z} dz + C$$

$$zU(z) = -\frac{e^{-\pi z}}{-\pi} + C \Rightarrow U(z) = \frac{e^{-\pi z}}{\pi z} + \frac{C}{z}$$

Luego se recupera $u(t)$ mediante la transformada inversa de Laplace

$$u(t) \equiv \mathcal{L}^{-1}[U(z)] = \lim_{R\to\infty} \frac{1}{2\pi i} \oint_{|z|=R} H(t) U(z) e^{zt} dz$$

$$u(t) = \mathcal{L}^{-1}[U(z)] = \frac{1}{\pi}\mathcal{L}^{-1}\left[\frac{e^{-\pi z}}{z}\right] + C\mathcal{L}^{-1}\left[\frac{1}{z}\right] = \frac{1}{\pi} H(t-\pi) + CH(t)$$

Se determina la constante con la condición inicial $u(0) = 1$:

$$u(t) = \frac{1}{\pi} H(t-\pi) + CH(t) \Rightarrow u(0) = \frac{1}{\pi} H(0-\pi) + CH(0) \Rightarrow C = u(0) = 1$$

$$u(t) = \frac{1}{\pi} H(t-\pi) + H(t)$$

PROBLEMA 22

Resuelva el siguiente problema de valores iniciales

$$\begin{cases} y'''(t) - y'(t) = 1 \\ y(0) = 0 \\ y'(0) = 0 \\ y''(0) = 1 \end{cases}$$

Reduciendo al problema de funciones causales.

SOLUCIÓN:

Sea $u(t) = H(t)y(t)$

$$u'_{gen}(t) = H(t)y'(t) + y(0)\delta(t)$$

$$u''_{gen}(t) = H(t)y''(t) + y(0)\delta'(t) + y'(0)\delta(t)$$

$$u'''_{gen}(t) = H(t)y'''(t) + y(0)\delta''(t) + y'(0)\delta'(t) + y''(0)\delta(t)$$

Se sustituye $y(0) = y'(0) = 0$ y $y''(0) = 1$ en las expresiones anteriores

$$u'_{gen}(t) = H(t)y'(t) + 0 \cdot \delta(t) = H(t)y'(t)$$

$$u'''_{gen}(t) = H(t)y'''(t) + 0 \cdot \delta''(t) + 1 \cdot \delta'(t) + y''(0)\delta(t) = H(t)y'''(t) + \delta(t)$$

Por lo tanto

$$u'''_{gen}(t) - u'_{gen}(t) = H(t)y'''(t) + \delta(t) - H(t)y'(t) = H(t)\big(y'''(t) - y'(t)\big) + \delta(t)$$

$$= H(t) \cdot 1 + \delta(t) = H(t) + \delta(t)$$

La ecuación en derivadas generalizadas es

$$u'''_{gen}(t) - u'_{gen}(t) = H(t) + \delta(t)$$

Sea $U: \mathbb{C} \to \mathbb{C}$ la transformada de Laplace de $u(t)$ definida por

$$U(z) \equiv \mathcal{L}[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-zt}dt$$

Se aplica la transformada de Laplace a cada término de la ecuación en derivadas generalizadas:

$$\mathcal{L}\big[\, u'''_{gen}(t)\big] - \mathcal{L}\big[u'_{gen}(t)\big] = \mathcal{L}[H(t)] + \mathcal{L}[\delta(t)]$$

$$\mathcal{L}[H(t)] = \frac{1}{z}$$

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}\big[\, u'_{gen}(t)\big] = zU(z)$$

$$\mathcal{L}[\,u'''_{gen}(t)] = z^3 U(z)$$

Sustituyendo cada uno de los términos:

$$z^3 U(z) - zU(z) = \frac{1}{z} + 1 \quad \Rightarrow \quad (z^3 - z)U(z) = \frac{z+1}{z} \quad \Rightarrow \quad U(z) = \frac{z+1}{z(z^3 - z)}$$

Donde $z = -1$ es una singularidad evitable:

$$U(z) = \frac{z+1}{z^2(z^2-1)} = \frac{z+1}{z^2(z-1)(z+1)} = \frac{1}{z^2(z-1)}$$

Luego se separa $U(z)$ en fracciones simples

$$U(z) = \frac{1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} = \frac{Az(z-1) + B(z-1) + Cz^2}{z^2(z-1)}$$

y se obtienen los coeficientes A, B, C :

$$A(z^2 - z) + B(z-1) + Cz^2 = 1 \quad \Rightarrow \quad \begin{cases} z^2: \; A + C = 0 \\ z^1: \; -A + B = 0 \\ z^0: \; -B = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} B = -1 \\ A = B = -1 \\ C = -A = -(-1) = 1 \end{cases}$$

Por lo tanto

$$U(z) = \frac{1}{z^2(z-1)} = -\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z-1}$$

Luego, se recupera $u(t)$ mediante la transformada inversa de Laplace definida por

$$u(t) \equiv \mathcal{L}^{-1}[U(z)] = \lim_{R\to\infty} \frac{1}{2\pi i} \oint_{|z|=R} H(t)U(z)e^{zt}dz$$

$$u(t) = \mathcal{L}^{-1}[U(z)] = -\mathcal{L}^{-1}\left[\frac{1}{z}\right] - \mathcal{L}^{-1}\left[\frac{1}{z^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{z-1}\right] = -H(t) - tH(t) + H(t)e^t$$

$$u(t) = H(t)(e^t - t - 1)$$

Como $u(t) = H(t)y(t)$ por lo tanto la función $y(t) \in \mathcal{C}^\infty(\mathbb{R})$ que satisface la ecuación es

$$u(t) = H(t)y(t) = H(t)(e^t - t - 1) \quad \Rightarrow \quad y(t) = e^t - t - 1 \quad \text{para} \;\; t \in \mathbb{R}$$

PROBLEMA 23

Resuelva el siguiente problema de valores iniciales

$$\begin{cases} ty''(t) + 2y'(t) + ty(t) = 2e^t \\ y(0) = -1 \\ y'(0) = -1 \end{cases}$$

Reduciendo al problema de funciones causales.

SOLUCIÓN:

Sea $u(t) = H(t)y(t)$

$$u'_{gen}(t) = H(t)y'(t) + y(0)\delta(t)$$

$$u''_{gen}(t) = H(t)y''(t) + y(0)\delta'(t) + y'(0)\delta(t)$$

Se sustituye $y(0) = -1$ y $y'(0) = -1$ en las expresiones anteriores

$$u'_{gen}(t) = H(t)y'(t) + (-1)\cdot\delta(t) = H(t)y'(t) - \delta(t)$$

$$u''_{gen}(t) = H(t)y''(t) + (-1)\cdot\delta'(t) + (-1)\cdot\delta(t) = H(t)y''(t) - \delta'(t) - \delta(t)$$

Por lo tanto

$$tu''_{gen}(t) + 2u'_{gen}(t) + tu(t) = t\big(H(t)y''(t) - \delta'(t) - \delta(t)\big) + 2\big(H(t)y'(t) - \delta(t)\big) + tH(t)y(t)$$

$$= H(t)\big(ty''(t) + 2y'(t) + ty(t)\big) - t\delta'(t) - t\delta(t) - 2\delta(t) = 2H(t)e^t - t\delta'(t) - t\delta(t) - 2\delta(t)$$

La ecuación en derivadas generalizadas es

$$tu''_{gen}(t) + 2\,u'_{gen}(t) + tu(t) = 2H(t)e^t - t\delta'(t) - t\delta(t) - 2\delta(t)$$

Sea $U: \mathbb{C} \to \mathbb{C}$ la transformada de Laplace de $u(t)$ definida por

$$U(z) \equiv \mathcal{L}[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-zt}dt$$

Se aplica la transformada de Laplace a cada término de la ecuación en derivadas generalizadas:

$$\mathcal{L}\big[t\,u''_{gen}(t)\big] + 2\mathcal{L}\big[u'_{gen}(t)\big] + \mathcal{L}[tu(t)] = 2\mathcal{L}[H(t)e^t] - \mathcal{L}[t\delta'(t)] - \mathcal{L}[t\delta(t)] - 2\mathcal{L}[\delta(t)]$$

Hallando la transformada de Laplace de cada término

$$\mathcal{L}[u(t)] = U(z)$$

$$\mathcal{L}[tu(t)] = -\frac{d}{dz}\big(U(z)\big) = -U'(z)$$

$$\mathcal{L}\big[u'_{gen}(t)\big] = zU(z)$$

$$\mathcal{L}\big[tu''_{gen}(t)\big] = -\frac{d}{dz}\big(z^2U(z)\big) = -2zU(z) - z^2U'(z)$$

$$\mathcal{L}[H(t)e^t] = \frac{1}{z-1}$$

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[t\delta(t)] = -\frac{d}{dz}(\mathcal{L}[\delta(t)]) = -\frac{d}{dz}(1) = 0$$

$$\mathcal{L}[\delta'(t)] = z\mathcal{L}[\delta(t)] = z$$

$$\mathcal{L}[t\delta'(t)] = -\frac{d}{dz}(\mathcal{L}[\delta'(t)]) = -\frac{d}{dz}(z) = -1$$

Sustituyendo cada uno de los términos por las respectivas transformadas de Laplace

$$-2zU(z) - z^2U'(z) + 2\big(zU(z)\big) + \big(-U'(z)\big) = 2\cdot\frac{1}{z-1} - (-1) - 0 - 2\cdot 1$$

$$-2zU(z) - z^2U'(z) + 2zU(z) - U'(z) = \frac{2}{z-1} - 1$$

$$-z^2U'(z) - U'(z) = \frac{2-(z-1)}{z-1} \qquad \Rightarrow \qquad -(z^2+1)U'(z) = \frac{3-z}{z-1}$$

Despejando $U'(z)$ de la ecuación anterior

$$U'(z) = \frac{z-3}{(z-1)(z^2+1)}$$

Se separa en fracciones simples

$$U'(z) = \frac{z-3}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1} = \frac{A(z^2+1)+(Bz+C)(z-1)}{(z-1)(z^2+1)}$$

Por lo que se deben hallar los valores de A, B y C tales que

$$A(z^2+1) + (Bz+C)(z-1) = z-3$$

Sustituyendo algunos valores de z se obtienen estos coeficientes

$$\begin{cases} z=1:\ A(1+1) = 1-3 \quad \Rightarrow \quad 2A = -2 \quad \Rightarrow \quad A = -1 \\ z=0:\ A(0+1)+(0+C)(0-1) = 0-3 \quad \Rightarrow \quad A - C = -3 \quad \Rightarrow \quad C = 2 \\ z=-1:\ A(1+1)+(-B+C)(-1-1) = -1-3 \quad \Rightarrow \quad 2A+2B-2C = -4 \quad \Rightarrow \quad B = 1 \end{cases}$$

Así

$$U'(z) = \frac{z-3}{(z-1)(z^2+1)} = -\frac{1}{z-1} + \frac{z}{z^2+1} + 2\frac{1}{z^2+1}$$

Luego, se recupera $u(t)$ mediante la transformada inversa de Laplace definida por

$$u(t) \equiv \mathcal{L}^{-1}[U(z)] = \lim_{R\to\infty}\frac{1}{2\pi i}\oint_{|z|=R} H(t)U(z)e^{zt}dz$$

Tomando transformada inversa de Laplace en la ecuación anterior

$$\mathcal{L}^{-1}[U'(z)] = -\mathcal{L}^{-1}\left[\frac{1}{z-1}\right] + \mathcal{L}^{-1}\left[\frac{z}{z^2+1}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{z^2+1}\right]$$

Donde

$$\mathcal{L}^{-1}[U'(z)] = -tu(t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{z-1}\right] = H(t)e^t$$

$$\mathcal{L}^{-1}\left[\frac{z}{z^2+1}\right] = H(t)\cos t$$

$$\mathcal{L}^{-1}\left[\frac{1}{z^2+1}\right] = H(t)\,\mathrm{sen}\,t$$

Así se tiene que

$$-tu(t) = -H(t)e^t + H(t)\cos t + 2H(t)\,\mathrm{sen}\,t$$

$$u(t) = \frac{H(t)e^t - H(t)\cos t - 2H(t)\,\mathrm{sen}\,t}{t} = H(t)\left(\frac{e^t - \cos t - 2\,\mathrm{sen}\,t}{t}\right)$$

Como $u(t) = H(t)y(t)$ entonces

$$y(t) = \frac{e^t - \cos t - 2\,\mathrm{sen}\,t}{t} \qquad \text{para} \quad -\infty < t < \infty$$

PROBLEMA 24

Resuelva el siguiente problema de valores iniciales

$$\begin{cases} t\dfrac{d^2y}{dt^2} + 2t\dfrac{dy}{dt} - 2y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Reduciendo al problema de funciones causales.

SOLUCIÓN:

Sea $u(t) = H(t)y(t)$

$$u'_{gen}(t) = H(t)y'(t) + y(0)\delta(t)$$

$$u''_{gen}(t) = H(t)y''(t) + y(0)\delta'(t) + y'(0)\delta(t)$$

Se sustituye $y(0) = 0$ y $y'(0) = 1$ en las expresiones anteriores

$$u'_{gen}(t) = H(t)y'(t) + 0 \cdot \delta(t) = H(t)y'(t)$$

$$u''_{gen}(t) = H(t)y''(t) + 0 \cdot \delta'(t) + 1 \cdot \delta(t) = H(t)y''(t) + \delta(t)$$

Por lo tanto

$$t\, u''_{gen}(t) + 2t\, u'_{gen}(t) - 2u(t) = t\big(H(t)y''(t) + \delta(t)\big) + 2tH(t)y'(t) - 2H(t)y(t)$$

$$= H(t)\big(t\, y''(t) + 2t\, y'(t) - 2y(t)\big) + t\delta(t) = H(t) \cdot 0 + t\delta(t) = t\delta(t)$$

La ecuación en derivadas generalizadas es

$$t\, u''_{gen}(t) + 2t\, u'_{gen}(t) - 2u(t) = t\delta(t)$$

Sea $U: \mathbb{C} \to \mathbb{C}$ la transformada de Laplace de $u(t)$ definida por

$$U(z) \equiv \mathcal{L}[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-zt}dt$$

Se aplica la transformada de Laplace a cada término de la ecuación en derivadas generalizadas:

$$\mathcal{L}\big[t\, u''_{gen}(t)\big] + 2\mathcal{L}\big[t\, u'_{gen}(t)\big] - 2\mathcal{L}[u(t)] = \mathcal{L}[t\delta(t)]$$

$$\mathcal{L}[u(t)] = U(z)$$

$$\mathcal{L}\big[t\, u'_{gen}(t)\big] = -\frac{d}{dz}\big(zU(z)\big) = -U(z) - zU'(z)$$

$$\mathcal{L}\big[t\, u''_{gen}(t)\big] = -\frac{d}{dz}\big(z^2U(z)\big) = -2zU(z) - z^2U'(z)$$

$$\mathcal{L}[t\delta(t)] = -\frac{d}{dz}(\mathcal{L}[\delta(t)]) = -\frac{d}{dz}(1) = 0$$

Sustituyendo cada uno de los términos:

$$-2zU(z) - z^2U'(z) + 2\big(-U(z) - zU'(z)\big) - 2U(z) = 0$$

$$-2zU(z) - z^2U'(z) - 2U(z) - 2zU'(z) - 2U(z) = 0$$

$$(-z^2 - 2z)U'(z) + (-2z - 2 - 2)U(z) = 0 \;\;\Rightarrow\;\; -z\,(z+2)U'(z) - 2(z+2)U(z) = 0$$

$$z\,U'(z) + 2U(z) = 0 \quad \Rightarrow \quad \frac{U'(z)}{U(z)} = -\frac{2}{z} \;\;\Rightarrow \frac{dU}{U} = -2\frac{dz}{z}$$

$$\int \frac{dU}{U} = -2\int \frac{dz}{z} \Rightarrow \ln U = -2\ln(Cz) \qquad \text{con} \quad C \in \mathbb{C} - \{0\}$$

$$\ln U = -2\ln(Cz) \quad \Rightarrow \quad \ln U = \ln(Cz)^{-2} \quad \Rightarrow \quad U(z) = \frac{1}{(Cz)^2} = \frac{K}{z^2}$$

Luego, se recupera $u(t)$ mediante la transformada inversa de Laplace definida por

$$u(t) \equiv \mathcal{L}^{-1}[U(z)] = \lim_{R\to\infty} \frac{1}{2\pi i} \oint_{|z|=R} H(t)U(z)e^{zt}dz$$

Esta integral se resuelve mediante el teorema de los residuos o se busca la antitransformada mediante teoremas operacionales de transformadas de Laplace

$$\mathcal{L}^{-1}[U(z)] = \mathcal{L}^{-1}\left[\frac{K}{z^2}\right] \quad \Rightarrow \quad u(t) = K\cdot(-1)tH(t)$$

$$u(t) = K\cdot(-1)tH(t) = H(t)y(t) \quad \Rightarrow \quad y(t) = -Kt$$

Y la constante K se encuentra usando una de las condiciones iniciales:

$$y'(t) = -K \;\;\Rightarrow\;\; K = -y'(0) = -1$$

Por lo tanto la función $y(t) \in C^\infty(\mathbb{R})$ que satisface la ecuación es

$$y(t) = t \qquad \text{para} \quad -\infty < t < \infty$$

PROBLEMA 25

Resuelva el problema de Cauchy

$$\begin{cases} \dfrac{d^2y(t)}{dt^2} + y(t) = t \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

donde $y(t) \in C^\infty(\mathbb{R})$ reduciendo al problema de funciones causales.

SOLUCIÓN:

Sea $u(t) = H(t)y(t)$

$$u'_{gen}(t) = H(t)y'(t) + y(0)\delta(t)$$

$$u''_{gen}(t) = H(t)y''(t) + y(0)\delta'(t) + y'(0)\delta(t) = H(t)y''(t) + 1 \cdot \delta'(t) + 1 \cdot \delta(t) =$$

$$= H(t)y''(t) + \delta'(t) + \delta(t)$$

Por lo tanto

$$u''_{gen}(t) + u(t) = H(t)y''(t) + \delta'(t) + \delta(t) + H(t)y(t)$$

$$= \left(y''(t) + y(t)\right)H(t) + \delta'(t) + \delta(t) = tH(t) + \delta'(t) + \delta(t)$$

La ecuación en derivadas generalizadas es

$$u''_{gen}(t) + u(t) = tH(t) + \delta'(t) + \delta(t)$$

Sea $U: \mathbb{C} \to \mathbb{C}$ la transformada de Laplace de $u(t)$ definida por

$$U(z) \equiv \mathcal{L}[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-zt}dt$$

Se aplica la transformada de Laplace a cada término de la ecuación en derivadas generalizadas:

$$\mathcal{L}\left[\, u''_{gen}(t)\right] + \mathcal{L}[u(t)] = \mathcal{L}[tH(t)] + \mathcal{L}[\delta'(t)] + \mathcal{L}[\delta(t)]$$

$$\mathcal{L}[u(t)] = U(z)$$

$$\mathcal{L}\left[\, u''_{gen}(t)\right] = z^2U(z)$$

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[\delta'(t)] = z$$

$$\mathcal{L}[tH(t)] = (-1)\frac{d}{dz}\left(\frac{1}{z}\right) = \frac{1}{z^2}$$

Sustituyendo cada uno de los términos

$$z^2U(z) + U(z) = \frac{1}{z^2} + z + 1$$

$$(z^2 + 1)U(z) = \frac{z^3 + z^2 + 1}{z^2} \quad \Rightarrow \quad U(z) = \frac{z^3 + z^2 + 1}{z^2(z^2 + 1)}$$

Luego se separa $U(z)$ en fracciones simples

$$U(z) = \frac{z^3 + z^2 + 1}{z^2(z^2 + 1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{Cz + D}{z^2 + 1} = \frac{Az(z^2 + 1) + B(z^2 + 1) + (Cz + D)z^2}{z^2(z - 1)}$$

$$A(z^3 + z) + B(z^2 + 1) + Cz^3 + Dz^2 = z^3 + z^2 + 1 \Rightarrow \begin{cases} z^3: \ A + C = 1 \quad \Rightarrow \quad C = 1 - A = 1 \\ z^2: \ B + D = 1 \quad \Rightarrow \quad D = 1 - B = 0 \\ z^1: \ A = 0 \\ z^0: \ B = 1 \end{cases}$$

Por lo tanto

$$U(z) = \frac{z^3 + z^2 + 1}{z^2(z^2 + 1)} = \frac{0}{z} + \frac{1}{z^2} + \frac{z + 0}{z^2 + 1} = \frac{1}{z^2} + \frac{z}{z^2 + 1}$$

Luego, se recupera $u(t)$ mediante la transformada inversa de Laplace definida por

$$u(t) \equiv \mathcal{L}^{-1}[U(z)] = \lim_{R \to \infty} \frac{1}{2\pi i} \oint_{|z|=R} H(t)U(z)e^{zt}dz$$

Tomando transformada inversa de Laplace en la ecuación anterior

$$\mathcal{L}^{-1}[U(z)] = \mathcal{L}^{-1}\left[\frac{1}{z^2}\right] + \mathcal{L}^{-1}\left[\frac{z}{z^2 + 1}\right] \quad \Rightarrow \quad u(t) = tH(t) + H(t)\cos t = H(t)(t + \cos t)$$

$$u(t) = H(t)(t + \cos t)$$

Como $u(t) = H(t)y(t)$ por lo tanto la función $y(t) \in \mathcal{C}^\infty(\mathbb{R})$ que satisface la ecuación es

$$u(t) = H(t)y(t) = H(t)(t + \cos t) \quad \Rightarrow \quad y(t) = t + \cos t \quad \text{para} \quad t \in \mathbb{R}$$

PROBLEMA 26

Sea $\mathcal{T}$ el operador diferencial definido por

$$\mathcal{T}=\frac{d^3}{dt^3}-\frac{d^2}{dt^2}+\frac{d}{dt}-\mathbb{I}$$

(a) Encuentre la solución fundamental del operador $\mathcal{T}$, es decir la función $u(t)\in\mathcal{C}^{\infty}(\mathbb{R})$ tal que

$$\mathcal{T}_{gen}u(t)=\delta(t)$$

(b) Usando la parte (a) resuelva el siguiente problema de valores iniciales

$$\begin{cases}\mathcal{T}w(t)=0\\ w(2)=0\\ w'(2)=-1\\ w''(2)=-1\end{cases}$$

SOLUCIÓN:

$$\mathcal{T}_{gen}u(t)=\delta(t)\Rightarrow\left(\frac{d^3}{dt^3}-\frac{d^2}{dt^2}+\frac{d}{dt}-\mathbb{I}\right)u(t)=\delta(t)$$

$$u_{gen}'''(t)-u_{gen}''(t)+u_{gen}'(t)-u(t)=\delta(t)$$

Y se aplicando tranformadas de Laplace en ambos lados de la ecuación anterior

$$\mathcal{L}\left[u_{gen}'''(t)-u_{gen}''(t)+u_{gen}'(t)-u(t)\right]=\mathcal{L}[\delta(t)]$$

$$(z^3-z^2+z-1)U(z)=1\quad\Rightarrow\quad U(z)=\frac{1}{z^3-z^2+z-1}=\frac{1}{(z-1)(z^2+1)}$$

Se aplica separación en fracciones simples a la función $U(z)$:

$$U(z)=\frac{1}{(z-1)(z^2+1)}=\frac{A}{z-1}+\frac{Bz+C}{z^2+1}=\frac{A(z^2+1)+(z-1)(Bz+C)}{(z-1)(z^2+1)}$$

$$1=A(z^2+1)+Bz^2+Cz-Bz-C$$

$$\begin{cases}z^2:\quad A+B=0\\ z^1:\quad C-B=0\\ z^0:\quad A-C=1\end{cases}\quad\Rightarrow\quad\begin{cases}A=1/2\\ B=-1/2\\ C=-1/2\end{cases}$$

$$U(z)=\frac{1/2}{z-1}-\frac{1}{2}\left(\frac{z+1}{z^2+1}\right)=\frac{1}{2}\cdot\frac{1}{z-1}-\frac{1}{2}\left(\frac{z}{z^2+1}+\frac{1}{z^2+1}\right)$$

Para recuperar la función $u(t)$ se utiliza la transformada inversa de Laplace

$$u(t)\equiv\mathcal{L}^{-1}[U(z)]=\mathcal{L}^{-1}\left[\frac{1}{2}\cdot\frac{1}{z-1}-\frac{1}{2}\left(\frac{z}{z^2+1}+\frac{1}{z^2+1}\right)\right]$$

$$=\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{z-1}\right]-\frac{1}{2}\mathcal{L}^{-1}\left[\frac{z}{z^2+1}\right]-\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{z^2+1}\right]=H(t)\left(\frac{1}{2}e^t-\frac{1}{2}\cos t-\frac{1}{2}\operatorname{sen}t\right)$$

(a) La solución fundamental del operador $\mathcal{T}$ es

$$u(t) = H(t)\left(\frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\operatorname{sen} t\right)$$

(b) Sea $f(t) = H(t-2)w(t)$. Para resolver el problema de valores iniciales, se calcula primero $\mathcal{T}_{gen}f(t)$:

$$f'_{gen}(t) = H(t-2)w'(t) + w(2)\delta(t-2) = H(t-2)w'(t)$$

$$f''_{gen}(t) = H(t-2)w''(t) + w(2)\delta'(t-2) + w'(2)\delta(t-2) = H(t-2)w''(t) - \delta(t-2)$$

$$f'''_{gen}(t) = H(t-2)w'''(t) + w(2)\delta''(t-2) + w'(2)\delta'(t-2) + w''(2)\delta(t-2)$$
$$= H(t-2)w'''(t) - \delta'(t-2) - \delta(t-2)$$

$$\mathcal{T}_{gen}f(t) = f'''_{gen}(t) - f''_{gen}(t) + f'_{gen}(t) - f(t) =$$

$$= H(t-2)w'''(t) - \delta'(t-2) - \delta(t-2) - \big(H(t-2)w''(t) - \delta(t-2)\big) + H(t-2)w'(t)$$
$$- H(t-2)w(t) =$$

$$= H(t-2)w'''(t) - \delta'(t-2) - H(t-2)w''(t) + H(t-2)w'(t) - H(t-2)w(t) =$$

$$= H(t-2)\big(w'''(t) - w''(t) + w'(t) - w(t)\big) - \delta'(t-2) =$$

$$= H(t-2)\mathcal{T}w(t) - \delta'(t-2) = 0 - \delta'(t-2) = -\delta'(t-2)$$

Se tiene que

$$\mathcal{T}_{gen}f(t) = -\delta'(t-2)$$

$$\mathcal{T}_{gen}u(t) = \delta(t) \Rightarrow \mathcal{T}_{gen}u(t) * f(t) = \delta(t) * f(t) \Rightarrow u(t) * \mathcal{T}_{gen}f(t) = f(t)$$

Se usa la fórmula de convolución para resolver el problema de valores iniciales a partir de la solución fundamental

$$f(t) = u(t) * \mathcal{T}_{gen}f(t) = u(t) * \big(-\delta'(t-2)\big) = -u'_{gen}(t) * \delta(t-2)$$

$$= -u'_{gen}(t-2) = -\frac{d}{dt}\left(H(t)\left(\frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\operatorname{sen} t\right)\right)\Bigg|_{t-2} =$$

$$\left(-H(t)\left(\frac{1}{2}e^t + \frac{1}{2}\operatorname{sen} t - \frac{1}{2}\cos t\right) - \left(\frac{1}{2}e^0 + \frac{1}{2}\operatorname{sen}(0) - \frac{1}{2}\cos(0)\right)\delta(t)\right)\Bigg|_{t-2} =$$

$$= H(t-2)\left(-\frac{1}{2}e^{t-2} - \frac{1}{2}\operatorname{sen}(t-2) + \frac{1}{2}\cos(t-2)\right)$$

La solución al problema de valores iniciales es

$$w(t) = -\frac{1}{2}e^{t-2} - \frac{1}{2}\operatorname{sen}(t-2) + \frac{1}{2}\cos(t-2) \qquad \text{con} \quad t \in \mathbb{R}$$